Design of safety distributed control under bounded time-varying communication delay

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Abstract— This paper addresses the design of a distributed safety controller for two agents, subject to communication delay. The control objective is to ensure safety, meaning that the state of the two-agent system does not enter an undesired set in the state space. Our approach is based on the computation of the complementary set to the maximal controlled invariant set, and on a state estimation procedure which guarantees control agreement between the two agents. We solve the safety control problem for any bounded communication delay, assuming that the two agents share the same internal clock. Performance of the controller and relevance of the proposed approach are discussed in light of simulations performed for a collision avoidance problem between two semi-autonomous vehicles at an intersection.

I. INTRODUCTION

The interest in coordinated control of partially or fully automated agents has recently surged with their diffusion in various engineering applications, from transport systems [9] to production lines [22]. Usually, in such configurations, the agents share information through a dedicated network and compute distributed inputs according to the received information. Such information can therefore be subject to communication delay.

In the past decade, the stability analysis of cooperative algorithms (i.e. consensus, rendezvous, flocking or synchronization problems) has attracted much attention in control theory (see [15] and [18] for recent reviews on the topic), but only a few works have investigated stability in the presence of communication or feedback delays. Pursuing the seminal work [16], most of these publications deal with constant delays (see [5], [13], [17] for example) which is seldom the case in practical applications and only very rare studies [3], [12], [23] consider the case of time-varying delays. In both cases, the core purpose of these approaches is to evaluate the *robustness* to delay of the delay-free algorithm or network topology [14].

In this paper, we investigate the design of a feedback methodology *dedicated to handle communication delays* for ensuring safety specifications. We consider the case of (potentially) partially automated agents which, due to the presence of a human operator in the loop, require to guarantee conflict-free behavior but in the least conservative manner. To address this problem, we follow a typical approach based on the computation of the complementary of the maximal controlled invariant set [9], [20]. The strategy

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that we advocate is grounded on the introduction of an additional controller ingredient, which is the construction of a (synchronized) estimated state set using delayed information. Counter-intuitively, to handle the distributed nature of the control, we do not employ here an estimation technique based on the most recent received data like it is done, for example, in [8] but voluntarily over-approximate the estimated set to guarantee that the agents employ the same information and agree on the control strategy to apply, which in turn guarantees safety. This is the main contribution of the paper.

For the sake of clarity, we only consider the simple case of two agents without measurement noises or uncertainties other than communication delays. However, the proposed approach can be naturally extended to N agents. The delay model under consideration is a non-continuous but bounded time-varying function. This framework allows, in particular, information reordering; other real-time effects such as dropout [7] or quantization [4] are not taken into account here. We prove that safety is guaranteed by the proposed synchronized estimation technique. Tuning of the proposed controller is also discussed and we show that closed-loop system performance decrease with the scale of the delay bound. The relevance of our approach is discussed in light of a simulation example.

The paper is organized as follows. We start by presenting the problem under consideration in Section II. Then, in Section III, we design safety control for the case of a bounded communication delay before providing an evaluation of the corresponding performance in Section IV. Finally, we apply the proposed technique on an example in Section V in which two vehicles negotiate an intersection to avoid collision.

Notation. In the following, m and p are positive integers. We denote with a superscript i the variables relatives to agent i for $i \in \{1, 2\}$ and with a subscript the coordinate. When the context is clear, for a vector $x = [z_1 \ldots z_m] \in \mathbb{R}^{\sum_{i=1}^{p} m_i}$ with $z_i \in \mathbb{R}^{p_i}$, we write $x_i = z_i$ for $i \in \{1, \ldots, m\}$.

$|.|$ denotes the Euclidean norm whereas $\|\cdot\|_\infty$ is used for the infinity norm of a signal. The diameter of a set S is written as $D(S) = \sup_{(x_1, x_2) \in S^2} |x_1 - x_2|$ and the distance between two sets S_1 and S_2 is written as $d(S_1, S_2) = \inf_{s_1 \in S_1, s_2 \in S_2} |s_1 - s_2|$. The boundary of a set S is written as δS and its closure as $\overline{S}$.

$\mathcal{C}^0_{pw}(S_1, S_2)$ represents the set of piecewise continuous functions defined on the set S_1 and taking values in S_2. $\mathcal{I}(\mathbb{R}^p)$ is the set of intervals included in $\mathbb{R}^p$. For two vectors x and $\tilde{x}$ in $\mathbb{R}^p$, we will write $x \leq \tilde{x}$ if $x_i \leq \tilde{x}_i$ for all $1 \leq i \leq p$. For $(\xi, \tilde{\xi}) \in \mathcal{C}^0_{pw}(S_1, S_2)^2$, we will write $\xi \leq \tilde{\xi}$ if $\xi(s) \leq \tilde{\xi}(s)$,

for all $s \in S_1$. For two vectors x and $\tilde{x}$ in $\mathbb{R}^p$, such that $x \leq \tilde{x}$, we write $[x, \tilde{x}] = [x_1, \tilde{x}_1] \times [x_1, \tilde{x}_2] \times \ldots \times [x_p, \tilde{x}_p]$.

$\varphi(t, t_0, x_0, u) \in \mathbb{R}^p$ is the flow associated with a given dynamics at time $t \geq t_0$ corresponding to the initial condition $x_0 \in \mathbb{R}^p$ at time $t_0 \geq 0$ driven by the input signal $u \in \mathcal{C}_{pw}^0([t_0, \infty), \mathbb{R}^m)$. For a set $S \subset \mathbb{R}^p$, we write $\varphi(t, t_0, S, u) = \cup_{x_0 \in S} \varphi(t, t_0, x_0, u)$. When possible, we will simply let $\varphi(t, S, u) = \varphi(t, 0, S, u)$. For $x : t \to \mathbb{R}^p$ and $0 \leq t_1 \leq t_2$, we write $x_{[t_1, t_2]} = \{x(s), \ s \in [t_1, t_2]\}$. When necessary, we write $\varphi(t, S, u_{[\bar{t}, \bar{t}+t)})$ the flow at time $t \geq 0$ driven by a portion of the input signal $u \in \mathcal{C}_{pw}([0, \infty), \mathbb{R}^m)$, with $\bar{t} \geq 0$.

A scalar continuous function $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if $\alpha(0) = 0$ and α is strictly increasing. A scalar continuous function $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and if $\alpha(t) \to \infty$ as $t \to \infty$.

II. PROBLEM STATEMENT

A. Agent dynamics

We consider that each agent obeys to the same dynamics, namely for $i \in \{1, 2\}$ we have

$$\dot{x}^i(t) = f^i(x^i(t), u^i(t)) \tag{1}$$

with $x^i \in \mathbb{R}^n$, $u^i \in [u_m, u_M] \subset \mathbb{R}^m$ and f^i satisfying the following property.

Assumption 1: For any initial condition $x_0 \in \mathbb{R}^n$ and any input $u \in \mathcal{C}_{pw}(\mathbb{R}_+, [u_m, u_M])$, the solution of (1) is global, unique and continuous with respect to the initial condition and the input.

It is worth noticing that this assumption also applies to the extended dynamics

$$\dot{x}(t) = \begin{pmatrix} f^1(x^1, u^1(t)) \\ f^2(x^2, u^2(t)) \end{pmatrix}, \tag{2}$$

in which $x = (x^1, x^2)$. In the sequel, we write $u = (u_1, u_2)$ and φ for the flow associated with (2).

B. Delay-free control design

Assumption 2: Given an open set $\mathcal{B} \subset \mathbb{R}^{2n}$, define

$$\mathcal{C} = \big\{ S \subset \mathbb{R}^{2n} \mid$$
$$\forall u \in \mathcal{C}_{pw}^0(\mathbb{R}_+, [u_m, u_M]^2) \ \exists t \geq 0 \quad \varphi(t, S, u) \cap \mathcal{B} \neq \emptyset \big\}.$$

Besides, define the operator

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^{2n} \times \mathcal{C}_{pw}(\mathbb{R}_+, 2^{[u_m, u_M]}) \to 2^{\mathbb{R}^{2n}}$$
$$(t, S, U) \mapsto \cup_{u \in U} \varphi(t, S, u). \tag{3}$$

There exists a non-decreasing feedback law $\pi : S_0 \subset \mathbb{R}^{2n} \mapsto 2^{[u_m, u_M] \times [u_m, u_M]}$ the values of which are Cartesian products in $\mathbb{R}^m \times \mathbb{R}^m$ and such that, provided that $S \notin \mathcal{C}$ and that $\tilde{\pi}(t) \subseteq \pi(\Phi(t, S, \tilde{\pi}_{[0, t)}))$ for $t \geq 0$, then $\Phi(t, S, \tilde{\pi}_{[0, t)}) \notin \mathcal{C}$, $t \geq 0$.

This assumption is a direct extension to the generalized flow Φ of the following more standard feedback definition:

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Cartesian products in $\mathbb{R}^m \times \mathbb{R}^m$ and such that, provided that $S \notin \mathcal{C}$ and that $u(t) \in \pi(\varphi(t, S, u_{[0, t)}))$ for $t \geq 0$, then $\varphi(t, S, u_{[0, t)}) \notin \mathcal{C}$, $t \geq 0$.

In general, it is possible to define π as a set-valued function, containing several feedback strategy alternatives. Yet, to facilitate the agreement strategy that the agents have to achieve, we voluntarily consider it as Cartesian product-valued. This point is crucial in the design of an agreement strategy. Due to this characteristic, one can refer without ambiguity to the ith dimension of the cross-product π as π_i. This is the notation we use later in the control strategy.

C. Agent communication and delays

We consider that communication delays occur between the two agents. Namely, defining $y^i(t)$ the information sent by agent i at time t and $Z^j(t)$ the set of information received by agent j at time t, we have

$$y^i(t) \in Z^j(t + \tau^j(t)) \tag{4}$$

in which $\tau^j \geq 0$ is a bounded delay, i.e., $\tau^j \in [0, \tau_M]$, $j = 1, 2$, with $\tau_M \geq 0$. Note that the set $Z^j(t)$ can be empty (if no information is received at time t) or can contain several elements (if more than one information is received at time t).

We assume that each agent knows its own entire state (from GPS measurements). Further, we consider that both agents share the same universal time t, obtained, e.g., from GPS measurements[1], and use it to stamp exchanged data. The information sent by the remote agent at time t is then

$$y^j(t) = (t, x^j(t)). \tag{5}$$

This implies that, for each exchanged information, the corresponding delay value is known, as the two agents can determine it by comparing the exchanged time stamp and the current time stamp.

D. Problem under consideration and proposed approach

The control objective is to guarantee safety, i.e., that the system state does not enter the bad set $\mathcal{B}$ at any time, despite communication delay. As the current communication delay is known, by comparison of the time stamps, a natural idea that arises is to estimate the current remote vehicle state by propagating the dynamics over a time interval of length equal to the current delay starting from a delayed measurement. However, as the remote agent inputs are unknown, the local agent can only obtain an interval of estimation of the remote vehicle state, while the remote vehicle knows its own state. Consequently, the two agents will evaluate the feedback map π defined in Assumption 2 on different sets[2]. Therefore, this strategy can cause the resulting applied controls to fail to guarantee safety.

In the sequel, we propose a strategy guaranteeing that the two vehicles are using the same set to evaluate the feedback map, which, in turn, guarantees safety.

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[1]We consider that the two agents are close enough so we can neglect the difference between the two received GPS signals [19].

[2]Further, as the values of the two delays are different a priori, one agent cannot know what information the other agent has received and uses as a starting point for propagation in the estimation. This causes the estimated set used by the remote agent to be unknown to the local agent.

III. CONTROL DESIGN FOR BOUNDED DELAY – AGREEMENT PROCEDURE

A. Synchronized delayed state

To counteract the effect of delay, a prediction strategy has to be employed in the control law. To guarantee safety, we propose to apply it to a delayed measurement common to both agents. We present this synchronized delayed state in what follows.

Introduce the delayed state corresponding to a measurement $z^i \in Z^i(t)$, $t \geq 0$, as

$$x_d^i(z^i) = \begin{pmatrix} x^1(z_1^1) \\ z_2^1 \end{pmatrix} \text{ if } i = 1 \text{ and } x_d^i(z^i) = \begin{pmatrix} z_2^2 \\ x^2(z_1^2) \end{pmatrix} \text{ if } i = 2 \tag{6}$$

in which $z_1^i = t - \tau$ $(i = 1,2)$ for a given $\tau \in [0, \tau_M]$, according to (4)–(5), and therefore $z_2^1 = x^2(t - \tau)$ and $z_2^2 = x^1(t - \tau)$. In details, each agent has access to the delayed state of the remote agent j $(j \neq i)$ from the second coordinate of the measurement $z_2^i = x^j(t - \tau)$ and, employing the measurement stamp $z_1^i = t - \tau$, can use its own corresponding state $x^i(x_1^i) = x^i(t - \tau)$ to compute $x_d^i(z^i)$.

We also consider the corresponding synchronized delayed state

$$x_d^{syn,i}(t) = \left\{ x_d^i(z^i) \mid t - z_1^i = \tau_M, \ z^i \in Z^i(s) \text{ and } s \leq t \right\}, \ t \geq \tau_M. \tag{7}$$

Lemma 1: The synchronized delayed state defined in (7) is such that $x_d^{syn,i}(t) \neq \emptyset$ for $t \geq \tau_M$. Further, $x_d^{syn,1}(t) = x_d^{syn,2}(t) = x(t - \tau_M)$ for $t \geq \tau_M$.

Proof: First, following (7), one can observe that showing that $x_d^{syn,i}(t) \neq \emptyset$ is equivalent to show that, for $i \in \{1,2\}$ and $t \geq \tau_M$, there exists $s \in [t - \tau_M, t]$ such that $z^i \in Z^i(s)$ and $t - z_1^i = \tau_M$. Consider $y^j(t - \tau_M)$, the information sent by the remote agent j at $t - \tau_M$. Following the delay definition (4), $y^j(t - \tau_M) \in Z^i(t_0)$ with $t_0 = t - \tau_M + \tau^i(t - \tau_M) \in [t - \tau_M, t]$. Therefore, there exists $z^i(t_0) \in Z^i(t_0)$ such that $z^i(t_0) = y^j(t - \tau_M)$ and, since $y_1^j(t - \tau_M) = t - \tau_M$ from (5), $z_1^i(t_0) = t - \tau_M$. Hence, there exists $t_0 \in [t - \tau_M, t]$ such that $t - z^i(t_0) = \tau_M$. Consequently, $x_d^{syn,i}(t)$ is not empty for $t \geq \tau_M$ and $i \in \{1,2\}$.

Second, from (7), $x_d^{syn,i}(t) = x_d^i(z^i)$ with $z_1^i = t - \tau_M$. Thus, using (5), one obtains

$$x_d^{syn,i}(t) = \begin{pmatrix} x^1(t - \tau_M) \\ x^2(t - \tau_M) \end{pmatrix} = x(t - \tau_M).$$

This concludes the proof. ∎

This lemma states that, when the delay is bounded, the synchronized state is well-defined as the information sent at time $t - \tau_M$ has been received at time t due to the fact that communication delay is upper-bounded by τ_M. Further, it expresses that, even if the computation of this state is agent-dependent, the state itself is the same for both agents. Therefore, in the sequel, we write $x_d^{syn} = x_d^{syn,1} = x_d^{syn,2}$.

B. Control design

According to Lemma 1, the two agents are able to compute the same delayed system state. Our approach is grounded on this synchronization technique, propagating this delayed state with the same input sets estimation set. This is the subject of the following theorem, in which this procedure gives rise to the definition of the estimated sets $\hat{x}^i(t)$.

Theorem 1: Consider the plant (1) satisfying Assumption 1, a feedback law π satisfying Assumption 2, the synchronized delayed state defined through (6)-(7) and the operator Φ defined in (3). Define, for $i \in \{1,2\}$,

$$\hat{x}^i(t) = \Phi(\tau_M, x_d^{syn,i}(t), U(i, \cdot)_{[t - \tau_M, t)}), t \geq \tau_M \tag{8}$$

$$U(i,t) = \begin{cases} \pi(\hat{x}^i(t)) & \text{if } t \geq \tau_M \\ [u_m, u_M]^2 & \text{otherwise} \end{cases}. \tag{9}$$

Provided that $x(0)$ is such that $\hat{x}^i(\tau_M) \notin \mathcal{C}$ for $i \in \{1,2\}$ and that $u^i(t) \in U_i(i,t)$ for $t \geq 0$ and $i \in \{1,2\}$, then, for $i \in \{1,2\}$,

$$x(t) \in \hat{x}^i(t), \ \hat{x}^i(t) \notin \mathcal{C} \text{ and } x(t) \notin \mathcal{B}, \ t \geq 0.$$

Proof: Using Lemma 1, we start by highlighting the fact that, for $t \geq \tau_M$, as $x_d^{syn,1}(t) = x_d^{syn,2}(t) = x_d^{syn}(t)$, then $\hat{x}^1(t) = \hat{x}^2(t) = \hat{x}(t)$ and $U(1,t) = U(2,t) = U(t)$. Further, for $t \geq 0$, $u(t) = (u^1(t), u^2(t)) \in U(1,t) \times U(2,t) = U(t)$, as π is Cartesian product-valued, according to Assumption 2.

Then, we show that $x(t) \in \hat{x}(t)$ for $t \geq \tau_M$. Using Lemma 1, one obtains that $\hat{x}(t) = \Phi(\tau_M, x(t - \tau_M), U_{[t - \tau_M, t)})$. Therefore, as $u(t) \in U(t)$ for $t \geq 0$, it follows that $x(t) = \varphi(\tau_M, x(t - \tau_M), u_{[t - \tau_M, t)}) \in \Phi(\tau_M, x(t - \tau_M), U_{[t - \tau_M, t)})$, $t \geq \tau_M$.

Now, it remains to show that $\hat{x}(t)$ never enters the capture set. By contradiction, consider that this does not hold, namely that there exists $t_1 > 0$ such that $\hat{x}(t_1) \in \mathcal{C}$ and define $t_0 = \sup\{t \in [0, t_1] \mid \hat{x}(t) \notin \mathcal{C}\}$, which exists by definition of the capture set and the fact that, by assumption, $\hat{x}(\tau_M) \notin \mathcal{C}$. By definition of the capture set, t_0 is such that $\hat{x}(t) \notin \mathcal{C}$ for $0 \leq t < t_0$ and $\hat{x}(t) \in \mathcal{C}$ for $t \in (t_0, t_0 + \delta)$ for a given $\delta > 0$. Consider $t_2 \in (t_0, t_0 + \min\{\tau_M, \delta\})$. Without loss of generality, in the following, we assume that $t_2 \geq 2\tau_M$. From what precedes, $x(t_2 - \tau_M) \in \hat{x}(t_2 - \tau_M, \mathcal{E}^{syn})$. Besides, as $u(t) \in U(t)$ for $t \geq 0$, we have, for $s \in [0, \tau_M)$,

$$\Phi(s, x(t_2 - \tau_M), U_{[t_2 - \tau_M, t_2 - \tau_M + s)})$$
$$= \Phi(s, \varphi(\tau_M - s, x(t_2 - 2\tau_M + s), u_{[t_2 - 2\tau_M + s, t_2 - \tau_M)}),$$
$$\quad U_{[t_2 - \tau_M, t_2 - \tau_M + s)})$$
$$\subseteq \Phi(s, \Phi(\tau_M - s, x(t_2 - 2\tau_M + s), U_{[t_2 - 2\tau_M + s, t_2 - \tau_M)}),$$
$$\quad U_{[t_2 - \tau_M, t_2 - \tau_M + s)})$$
$$\subseteq \Phi(\tau_M, x(t_2 - 2\tau_M + s), U_{[t_2 - 2\tau_M + s, t_2 - \tau_M + s)}) = \hat{x}(t_2 - \tau_M + s). \tag{10}$$

Further, from (8), $\Phi(\tau_M, x(t_2 - \tau_M), U_{[t_2 - \tau_M, t_2)}) = \hat{x}(t_2)$.

Moreover, the feedback law is non-decreasing, according to Assumption 2. Therefore, from (10), $\pi(\hat{x}(t_2 - \tau_M + s)) \subseteq \pi(\Phi(s, x(t_2 - \tau_M), U_{[t_2 - \tau_M, t_2 - \tau_M + s)}))$ for $0 \leq s \leq \tau_M$, and, consequently, $U(t_2 - \tau_M + s) \subseteq \pi(\Phi(s, x(t_2 - \tau_M), U_{[t_2 - \tau_M, t_2 - \tau_M + s)}))$ for $0 \leq s \leq \tau_M$. As

$x(t_2 - \tau_M) \in \hat{x}(t_2 - \tau_M) \notin \mathcal{C}$ because $t_2 - \tau_M < t_0$, then $\Phi(s, x(t_2 - \tau_M), U_{[t_2-\tau_M, t_2-\tau_M+s)}) \notin \mathcal{C}$ for $s \in [0, \tau_M]$, by Assumption 2. Therefore, in particular, for $s = \tau_M$, $\Phi(\tau_M, x(t_2 - \tau_M), U_{[t_2-\tau_M, t_2)}) = \hat{x}(t_2) \notin \mathcal{C}$. As $t_2 \in (t_0, t_0 + \delta)$, this is in contradiction with the fact that $\hat{x}(t) \in \mathcal{C}$ for $t \in (t_0, t_0 + \delta)$. Therefore, $\hat{x}(t) \notin \mathcal{C}$, $t \geq 0$. Consequently, as $x(t) \in \hat{x}(t)$, $x(t) \notin \mathcal{B}$ for $t \geq 0$. ∎

Theorem 1 provides both an invariance property and a safety result, exploiting the synchronized delayed state introduced in the previous section, which can be computed by both agents.

In order to further understand the meaning of the estimation set (8), note that, for each agent, it is possible to sharpen the state estimation by employing the most recent received measurement in lieu of $x_d^{syn}(t)$ in the propagated flow (8). However, with such a technique, one cannot guarantee that the two agents employ the same information to compute the feedback map, as the most recent measurement has no reason to be the same for both. In other words, $\hat{x}^1$ is different a priori from $\hat{x}^2$ and therefore $U(1, \cdot)$ can be different from $U(2, \cdot)$, which in turn, through propagation, implies that the future values of $\hat{x}^1$ and $\hat{x}^2$ are not coincident. This is why we propose to instead employ $\hat{x}_d^{syn}(t)$ in the control strategy.

IV. EVALUATION OF CLOSED-LOOP PERFORMANCE

In this section, we aim at providing an evaluation of the performance which can be obtained using the proposed technique, i.e., to compare how far from the bad set the trajectory generated by the control law can be with communication delay. With this aim in view, we define $\mathcal{U} = \{ u \in \mathcal{C}_{pw}(\mathbb{R}_+, [u_m, u_M]^2) \mid u_i(t) \in U_i(i, t), \, t \geq 0 \}$ and consider the following quantity

$$\inf_{t \geq 0} \sup_{u \in \mathcal{U}} \; d(\varphi(t, x_0, u), \mathcal{B}),$$

for given $x_0 \in \mathbb{R}^{2n}$ such that $\Phi(\tau_M, x(0), [u_m, u_M]^2) \notin \mathcal{C}$. This distance quantifies, for given initial conditions satisfying the assumptions of Theorem 1, how far from the bad set the trajectory generated by the proposed control law can be in the presence of communication delay. To bound this term, we first characterize further the dynamics under consideration.

Assumption 3: There exist a positive non-decreasing scalar continuous function κ, class $\mathcal{K}$ function α_1 and α_2 and a class $\mathcal{K}_\infty$ function γ such that, for arbitrary pairs $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(u_1, u_2) \in \mathbb{R}^m \times \mathbb{R}^m$, the corresponding solutions of (1) satisfy, for $t \geq 0$,

$$|\varphi(t, x_1, u_1) - \varphi(t, x_2, u_2)|$$
$$\leq \kappa(t)\alpha_1(|x_1 - x_2|) + \alpha_2(t)\gamma(\|u_2 - u_1\|_\infty). \quad (11)$$

This assumption is motivated by Theorem 3.4 in [11]. Indeed, provided that the vector field is continuously differentiable with respect to the state x and the input u, Assumption 3 holds. Therefore, it encompasses a large class of dynamical systems. It holds for example for incrementally input-to-state stable systems [1].

Proposition 1: Consider the plant (1) satisfying Assumptions 1 and 3, a feedback law π defined in Assumption 2, the operator Φ defined in (3) and the estimated

set defined through (6)-(7). Then, there exist a class $\mathcal{K}$ function α and a class $\mathcal{K}_\infty$ function γ such that, for given $x_0 \in \mathbb{R}^{2n}$ such that $\Phi(\tau_M, x(0), [u_m, u_M]^2) \notin \mathcal{C}$ and with $\mathcal{U} = \{ u \in \mathcal{C}_{pw}(\mathbb{R}_+, [u_m, u_M]^2) \mid u_i(t) \in U_i(i, t), \, t \geq 0 \}$, then

$$\inf_{t \geq 0} \sup_{u \in \mathcal{U}} \; d(\varphi(t, x_0, u), \mathcal{B})$$
$$\leq \inf_{t \geq 0} \sup_{u \in \mathcal{U}} \; d(\hat{x}^i(t), \mathcal{B}) + \alpha(\tau_M)\gamma(|u_M - u_m|). \quad (12)$$

Proof: In the sequel, we write $\hat{x} = \hat{x}^1(t) = \hat{x}^2(t)$, as highlighted previously. First, as $\varphi(t, x_0, u) \in \hat{x}(t)$, as proved in Theorem 1, it is possible to obtain the following bound

$$d(\varphi(t, x_0, u), \mathcal{B}) \leq D(\hat{x}(t)) + d(\hat{x}(t), \mathcal{B}). \quad (13)$$

By definition of the estimated sets (6)-(7) and of the input sets (9), there exists $\tilde{x}_0 \in \mathbb{R}^{2n}$ such that

$$D(\hat{x}(t)) \leq$$
$$\left| \max_{u:\mathbb{R}_+ \mapsto [u_m, u_M]^2} \varphi(\tau_M, \tilde{x}_0, u) - \min_{u:\mathbb{R}_+ \mapsto [u_m, u_M]^2} \varphi(\tau_M, \tilde{x}_0, u) \right|$$
$$\leq \alpha(\tau_M)\gamma(|u_M - u_m|), \quad (14)$$

in which we have applied (11) and in which α_2 is a class $\mathcal{K}$ function and γ is a class $\mathcal{K}_\infty$ function. Bounding (13) with (14) and taking the inf and sup of both sides for $t \geq 0$, the result follows, defining the class $\mathcal{K}$ function $\alpha = \alpha_2$. ∎

The closed-loop performance bound proposed in (12) involves a first term which only depends on the choice of the nominal feedback map and a second term which is increasing with τ_M and $|u_M - u_m|$. Then, this result states predictably that the conservativeness of the proposed approach increases with the scale of the maximum delay τ_M. However, one cannot reduce this conservativeness without compromising safety. Note that, for $\tau_M = 0$, i.e., without delay, the left-hand side is then equal to the right-hand side as $\hat{x}(t) = \varphi(t, x_0, u)$ and as the second term in the right-hand side of (12) is equal to zero. This bodes well for the tightness of this upper-bound.

V. NUMERICAL SIMULATIONS

In this section, we illustrate the merits of our approach by applying the proposed control strategy to a two-vehicle collision avoidance problem. We consider two human-driven vehicles approaching the traffic intersection depicted in Fig. 1. The vehicles are equipped with GPS
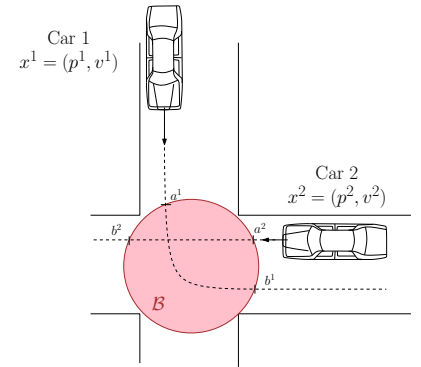

Fig. 1: The considered intersection. A collision occurs if the two vehicles are in the bad set $\mathcal{B}$ at the same time.

and exchange their respective state via Vehicle-to-Vehicle

(V2V) communication. The control objective is to guarantee that the two vehicles do not enter simultaneously the intersection despite communication delay.

A. Vehicle dynamics, delay-free feedback map and efficient computation of the proposed algorithm

For vehicle $i \in \{1,2\}$, we denote (see Fig. 1) the longitudinal displacement along its path by $x_1^i = p_i$ and the longitudinal speed by $x_2^i = v_i$. The considered longitudinal dynamics for vehicle $i \in \{1,2\}$ are

$$\dot{x}_1^i = x_2^i, \quad \dot{x}_2^i = \begin{cases} au + b & \text{if } x_2^i \in (v_m, v_M) \\ 0 & \text{otherwise}, \end{cases}$$

in which the input $u \in [0, u_M]$ with $u_M > 0$, $a > 0$ and $b < 0$ are given constants. This model is obtained employing Newton's law, assuming that the road is flat, and that the air drag term is negligible [21]. Finally, saturation on the velocity is employed to account for the fact that the vehicle speed is constrained between $v_m > 0$ and $v_M > 0$, i.e., the vehicle is not allowed to stop (to prevent the trivial solution in which the vehicles come to a stop) and from exceeding a maximum speed (to respect road speed limitations).

One can check that these dynamics satisfy Assumptions 1 and 3 and is monotone[3].

The bad set consists of two path portions the vehicles cannot be located at the same time, i.e., $\mathcal{B} = (a^1, b^1) \times (a^2, b^2)$. The monotonicity of the dynamics along with the structure of the bad set can be exploited to reformulate the capture set[4] in terms of restricted sets [6]. With this aim in view, consider a constant input $u \in [u_m, u_M]$, the corresponding restricted capture set $\mathcal{C}_u = \{x \in \mathbb{R}^{2n} \mid \exists t \geq 0 \; \varphi(t, x, u) \in \mathcal{B}\}$ and define the constant input signals $u_L = (u_M, u_m)$ and $u_H = (u_m, u_M)$. Then, we have, following [10],

$$\mathcal{C} = \{S \subset \mathbb{R}^{2n} \mid S \cap \mathcal{C}_{u_L} \neq \emptyset \text{ and } S \cap \mathcal{C}_{u_H} \neq \emptyset\},$$

and a feedback control map satisfying Assumption 2 is

$$\pi(S) = \begin{cases} u_H & \text{if } S \cap \mathcal{C}_{u_L} \neq \emptyset \text{ and } S \cap \partial \mathcal{C}_{u_H} \neq \emptyset \\ u_L & \text{if } S \cap \partial \mathcal{C}_{u_L} \neq \emptyset \text{ and } S \cap \overline{\mathcal{C}_{u_H}} \neq \emptyset \\ [u_m, u_M] \times [u_m, u_M] & \text{otherwise}. \end{cases}$$

Therefore, Theorem 1 and Proposition 1 hold. Further, as the dynamics under consideration is monotone, it is possible to compute the proposed control algorithm in an efficient manner. Indeed, in general, the presented control strategy can be computationally demanding, as it requires at each time step the computation of the operator Φ, given in (3) and employed in (8), and thus requires to explore the entire control set. However, for monotone dynamics, (8) can be calculated for closed-valued signals U as

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^{2n} \times \mathcal{C}_{pw}(\mathbb{R}_+, \mathcal{I}([u_m, u_M]^2)) \to 2^{\mathbb{R}^{2n}}$$
$$(t, x, U) \mapsto [\varphi(t, x, s \geq 0 \mapsto \min U(s)),$$
$$\varphi(t, x, s \geq 0 \mapsto \max U(s))].$$

[3] We say that the flow of (1) is monotone with respect to the input and the initial condition [2] if $\forall t \in \mathbb{R}_+ \; \forall (x, \tilde{x}) \in \mathbb{R}^n \times \mathbb{R}^n \; \forall (u, \tilde{u}) \in \mathcal{C}_{pw}^0(\mathbb{R}_+, [u_m, u_M])^2 \quad x \leq \tilde{x}, \; u \leq \tilde{u} \Rightarrow \varphi^i(t, x, u) \leq \varphi^i(t, \tilde{x}, \tilde{u})$.
[4] and to compute the corresponding feedback law with an algorithm of linear complexity with respect to the state dimension.
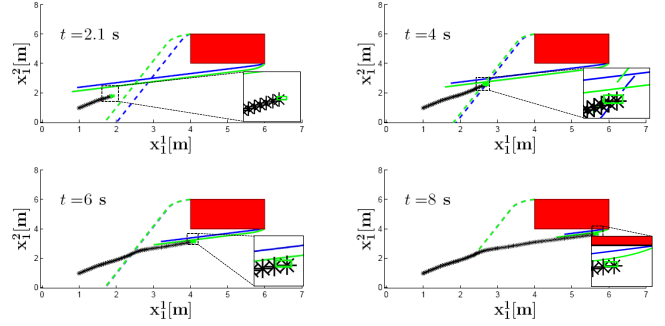
(a) Dynamic evolution of the closed-loop system on the (x_1^1, x_1^2) plane. The black asterisks represent the trajectory of the system projected onto the (x_1^1, x_1^2) plane. The red box represents the projection of $\mathcal{B}$, slices of the restricted capture set $\mathcal{C}_{u_L}$ (solid line) and $\mathcal{C}_{u_H}$ (dashed line) corresponding to the current speeds are pictured in blue. Slices corresponding to the current speed estimate (common to Vehicle 1 and Vehicle 2) are pictured in green and green rectangles represent the projection of $\hat{x}(t)$ on the (x_1^1, x_1^2) plane. Zooms in the current state and estimate set projections are provided on the bottom right corner of each plot.
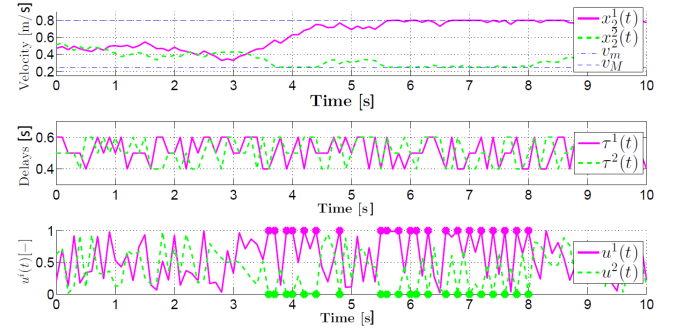
(b) Plots of the speeds, communication delays and control inputs (in magenta for Vehicle 1 and in green for Vehicle 2). The inputs corresponding to automatic control of the vehicles ($U(t) = \{u_L\}$ or $U(t) = \{u_H\}$) are represented with dots on the bottom plot.

Fig. 2: Simulation results for a communication delay with uniform distribution.

In particular, for the example considered here, all sets can be computed using only upper and lower bounds, propagated with extremal control values.

In the sequel, when $\pi(S) \subset [u_m, u_M]^2$, we say that automatic control of the vehicles is taken, meaning that the drivers are overridden and cannot control the cars.

B. Simulation results

For simulation, we choose $a = 1$, $b = -0.5$, $u_m = 0$, $u_M = 1$, with $v_m = .25$ m/s and $v_M = .8$ m/s. The bad set is $\mathcal{B} =]4, 6[\times \mathbb{R} \times]4, 6[\times \mathbb{R}$. We consider a discrete-time implementation, with a time step $\Delta T = .1$ s, and a discrete communication delay generated using a discrete uniform distribution on the discrete interval $\{0.4, 0.5, 0.6\}$ in seconds. The initial positions are generated randomly on the interval $[0.6, 1.2] \times [0.6, 1.2]$ and the initial speeds are both 0.5 m/s. The driver behavior follows a uniform distribution on the interval $[u_m, u_M]$. Consistently, we choose $\tau_M = 0.6$ s.

Simulation results are pictured in Fig. 2. Dynamic evolution of the system in the (x_1^1, x_1^2) plane is given in Fig. 2(a). One can observe that, as previously claimed, the estimation set $\hat{x}^i(t)$, $i = 1, 2$, computed by the two vehicles are equal. The essence of the control strategy is visible in Fig. 2(a): the proposed controller guarantees that the estimation set does
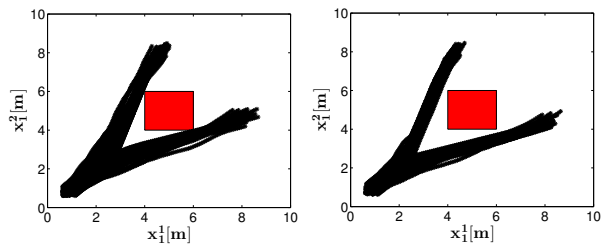
Fig. 3: 200 trajectories generated for a bounded communication delay with $\tau_M = 0.6$ s (left) and $\tau_M = 1.1$ s (right): black asterisks represent trajectories generated from random initial conditions, projected onto the (x_1^1, x_1^2) plane, and the red box represents the projection of $\mathcal{B}$.

not belong to the capture set. More generally, despite the delay appearance, safety is guaranteed. This is illustrated in Fig. 3 in which trajectories obtained for various initial conditions and with the proposed control algorithm are pictured. None of these trajectories intersects the bad set.

Finally, one can observe in Fig. 2(a) that the slice $\mathcal{C}_{u_L}$ corresponding to the estimate speeds is not coincident with the one corresponding to the actual speeds. Therefore, automatic control is applied before actually reaching the capture set: delay appearance is responsible of performance degradation. Fig. 3 pictures different trajectories generated with discrete uniform delay distribution, respectively on the interval $\{0.4, 0.5, 0.6\}$ and on $\{0.9, 1, 1.1\}$ in seconds and with the same parameters as previously. One can observe that the distance of the trajectories to the bad set is greater for the second case: system performance worsens with the maximum delay magnitude, which is consistent with Proposition 1.

VI. CONCLUSION

In this paper, we have addressed distributed safety control for two agents, subject to communication delay, by designing a state estimation procedure which guaranteed control agreement between the two agents. Resulting performance has also been investigated. Simulation results on an intersection collision avoidance problem for two semi-autonomous vehicles highlighted the interest of the proposed approach.

The requirement of simultaneous computation of the estimation set can seem quite fragile from an implementation point of view. Robustness of this control strategy to missynchronization between the two agents is a point of current investigation.

Besides, we use the maximum delay to compute the synchronized estimation set and guarantee safety. However, as the closed-loop system performance are scaled by τ_M as shown, this can result in conservative performance if, for example, the probability of obtaining the maximum delay value can be neglected. This is why it could be of interest to introduce an additive parameter $\tau < \tau_M$ in lieu of τ_M to compute the synchronized sets. In this case, of course, the synchronized set may not exist at all time and safety properties can only be obtained under the form of probabilities. Intuitively, this probability of safety will increase with τ. Therefore, this naturally lead to the realization that

there exists a fundamental tradeoff between performance degradation and safety. This is a direction of future work.

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