2004 Conference on Decision and Control (CDC)
http://www.cds.caltech.edu/~murray/papers/2004c_dm04a-cdc.html

Existence of Discrete State Estimators for Hybrid Systems on a Lattice

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***Abstract*— The observability properties of a class of hybrid systems whose continuous variables are available for measurement are considered. We show that the discrete variable dynamics can be always extended for observable systems to a lattice in such a way that the extended system has the properties that allow the construction of the LU discrete state estimator. It updates two variables at each step, the upper and lower bound of the set of all possible discrete variable values compatible with the output sequence. Causes of the estimator complexity are investigated.**

I. INTRODUCTION

The problem of estimating discrete variables in a class of hybrid systems whose continuous variables are available for measurement is considered. The continuous variables may represent physical quantities such as position and velocity, while the discrete variables may represent the state of an internal logical system or communications protocol used for communication and coordination as it happens in multi-robot systems.

The problem of estimating and tracking the values of non-measurable variables as well as the problem of studying observability properties in hybrid systems has been investigated by several authors. Bemporad et al. [3] show that observability properties are hard to check for hybrid systems and an observer is proposed that requires large amounts of computation. A wealth of research has been done on designing observers for discrete event systems both deterministic and non-deterministic. For non-deterministic systems, [9] studies observability conditions for exact reconstruction of the current state after each system event, and [5] consider the problem of finding optimal control strategies for partially observable Markov-decision processes. In the deterministic case, [4] and [7] show that the complexity of the observer often arises from the need to compute maps on large sets of values, corresponding to the set of all possible internal states compatible with the observed output sequence. Similar difficulties are encountered in [10], where an observer is proposed for a class of hybrid systems, that fails to be applicable for large problem sizes. In the models that we consider, due to the heavy coupling of discrete and continuous variables evolution, discrete state estimation strategies where the analysis of the continuous signal can suffice for determining the discrete state, such as the ones proposed by [1], are not applicable. Also, the time scales of the continuous and discrete dynamics are comparable. Thus, there is no guarantee that the system remains in each discrete mode for a sufficiently long time as assumed for example in [2] or [12], where the observability of discrete-time linear switched systems is characterized by assuming a minimum dwell-time between consecutive switches.

In [11] some of the complexity issues, such as those encountered in [10], [4], or [7], were overcome by finding a good way of representing the sets of interest and of computing maps on them. In particular, a system defined on a space of variables, was extend to a larger space of variables equipped with lattice structure to obtain an extended system. Provided this extended system satisfy certain requirements, a discrete state estimator, that we will refer to as LU estimator, can be constructed, which updates two variables at each step. It updates the lower (L) and upper (U) bound of the set of all values of discrete variables compatible with the output sequence and with the dynamics of the system. When is it possible to find a lattice structure such that the extended system satisfies the properties needed for the construction of such an estimator? In this paper we answer this question.

The main contribution of this paper is as follows. We show that a system is observable if and only if there is a lattice where the extended system satisfies the requirements for the construction of the LU estimator proposed in [11]. This shows that our approach to estimation is general. If the discrete variable dynamics is driven by an automaton, we show that our estimator and the one in [4] are comparable from a computational point of view. The main advantage of the lattice approach for estimation comes when the system enjoys order preserving properties on a lattice whose order relations can be computed algebraically, as is in the case of the multi-robot example proposed in [11]. In such an example, the computation of our estimator scales with the number of variables to be estimated rather than with the size of the space of discrete variables, which is typically huge in distributed systems.

There is a number of applications in which the discrete state dynamics evolves naturally on partially ordered sets. Resource allocation problems involving moving resources (agents) as in the case of air traffic controlled systems, weapon-target assignment problems are examples where the tasks are usually associated with positions in Euclidean space, where the usual cone partial order induces a partial order on the tasks. In the case of dynamic scheduling for distributed computing, the set of processes that need

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to be allocated to resources is typically partially ordered according to priorities, and the allocation to resources is dynamically established on the basis of such partial ordering. In addition, there is plenty of systems where a partial order among events is naturally established by a causal order relation, as in the case of message passing based distributed systems. Most of these examples are also distributed, meaning that the size of the discrete state is so large to render prohibitive the estimation problem if the partial order is not explicitly taken into account in the estimator design.

The contents of the paper are as follows. In Section II we review basic definitions on transition systems and basic notions on partial order theory. Section III gives the main result on the existence of a lattice. Section IV provides some complexity considerations. In Section V, we propose two examples.

II. BASIC DEFINITIONS

In this section we review basic notions on state transition systems, as these systems provide a general framework for modeling hybrid systems, and we review basic definitions from partial order and lattice theory, which will be used throughout the paper.

A. State Transition Systems

The basic definitions given here can be found in more detail in other work [8]. Let S be the set of states with $s \in S$. A *transition function* on S is a function $F : S \longrightarrow S$ which updates the state s to a new state $s' = F(s)$. Given a transition function F, an *execution* of F is a sequence $\sigma = \{s(k)\}_{k\in\mathbb{N}}$ such that $s(k+1) = F(s(k))$ for all $k \in \mathbb{N}$. The set of all executions of F is denoted $\mathscr{E}(F)$.

Definition 2.1: Let F be a transition function on a set of states S, the set $\Omega \subset S$ is the *ω-limit set* of F, denoted $\omega(F)$, if it is the smallest set such that the following hold:
(i) if $s \in \Omega$ and $s' \in F(s)$, then $s' \in \Omega$;
(ii) for each $\sigma \in \mathscr{E}(F)$, there exists a time k_σ such that $\sigma(k_\sigma) \in \Omega$ for all $k \geq k_\sigma$.

We now recall the notion of observability for transition systems as it can be found in [10].

Definition 2.2: (Observability) The transition function F is said to be *observable* with respect to the output function $g : S \to \mathscr{Y}$ if for any two different executions $\sigma_1, \sigma_2 \in \mathscr{E}(F)$ there exists a k such that $g(\sigma_1(k)) \neq g(\sigma_2(k))$.

We will consider transition systems where $s = (\alpha, z)$, where $\alpha \in \mathscr{U}$ is the discrete part of the state with $\mathscr{U}$ a finite discrete set, and $z \in \mathscr{Z}$ is the continuous portion of the state, for example $\mathscr{Z} = \mathbb{R}^n$. In such a case $F = (f, h)$, where $f : \mathscr{U} \times \mathscr{Z} \to \mathscr{U}$ is the function that updates the discrete state, and $h : \mathscr{U} \times \mathscr{Z} \to \mathscr{Z}$ is the function that updates the continuous part of the state. We will denote such transition systems by the tuple $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$. The executions of Σ, denoted $\mathscr{E}(\Sigma)$, are of the form $\sigma = \{\alpha(k), z(k)\}_{k\in\mathbb{N}}$, with $\alpha(k+1) = f(\alpha(k), z(k))$ and $z(k+1) = h(\alpha(k), z(k))$. Since we assume that the continuous variables z are measured, the output sequence is given by $g(\sigma) = \{y(k)\}_{k\in\mathbb{N}} := \{z(k)\}_{k\in\mathbb{N}}$.

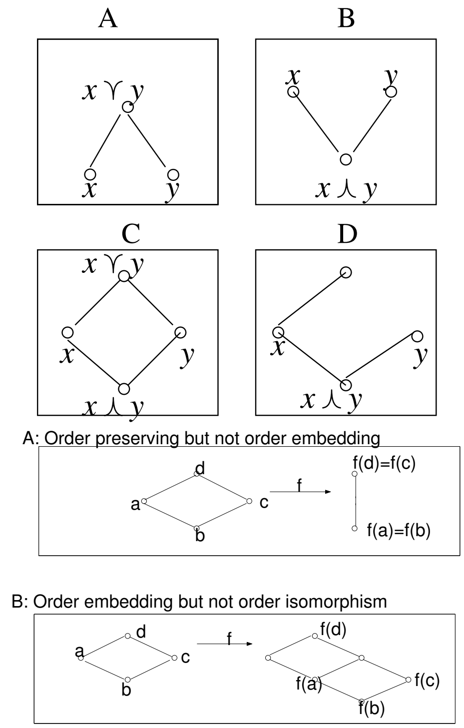

Fig. 1. (Up) In diagram A and B x and y are not related, but they have a join and a meet. In diagram C we show a complete lattice, and in diagram D we show an ordered set that is not a lattice, since the elements x and y have a meet, but not a join. (Down) In diagram A we show a map that is order preserving but not order embedding. In diagram B we show an order embedding map that is not order isomorphism: any two elements maintain the same order relation, but in between c and d there is nothing, while in between $f(c)$ and $f(d)$ some other elements appears (i.e. it is not onto).

B. Partial Order Theory

The definitions given here can be found in more detail in [6]. Given a set $(\chi, \leq)$ with an order relation "$\leq$", we define the *join* "$\curlyvee$" and the *meet* "$\curlywedge$" of two elements x and w in χ as $x \curlyvee w = \sup\{x, w\}$ and $x \curlywedge w = \inf\{x, w\}$. If $S \subseteq \chi$, $\bigvee S = \sup S$ and $\bigwedge S = \inf S$, where by $\sup\{x, w\}$ we denote the smallest element in $(\chi, \leq)$ that is larger than both x and w, and we denote by $\inf\{x, w\}$ the biggest element in $(\chi, \leq)$ that is smaller than both x and w. We denote by $|S|$ the cardinality of $S \subseteq \chi$. Let $(\chi, \leq)$ be a non-empty ordered set. If $x \curlywedge w$ and $x \curlyvee w$ exist for any $x, w \in \chi$, then $(\chi, \leq)$ is a *lattice*. If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq \chi$, then $(\chi, \leq)$ is a *complete lattice*. Notice that any finite lattice is complete. In Figure 1 (up) we report Hasse diagrams showing ordered sets.

Let $(\chi, \leq)$ be an ordered set. Then $(\chi, \leq)$ is a *chain* if for all $x, w \in \chi$, either $x \leq w$ or $w \leq x$, that is any two elements are comparable. At the opposite extreme of a chain is an antichain. The ordered set $(\chi, \leq)$ is an *antichain* if $x \leq y$ if and only if $x = y$. Let $(\chi, \leq)$ be a lattice and let $\emptyset \neq S \subseteq \chi$ be a subset of χ. Then S is a *sublattice* of $(\chi, \leq)$ if $a, b \in S$ implies that $a \curlyvee b \in S$ and $a \curlywedge b \in S$. Given a complete lattice $(\chi, \leq)$, we will be concerned with a special kind of a sublattice called an *interval sublattice* defined as follows. Any interval sublattice of $(\chi, \leq)$ is given by $[L,U] = \{w \in \chi : L \leq w \leq U\}$ for $L, U \in \chi$. That is, this special sublattice can be represented by only two elements. The *power lattice* of a set $\mathscr{U}$, denoted $(\mathscr{P}(\mathscr{U}), \subseteq)$, is given by the power set of $\mathscr{U}$, $\mathscr{P}(\mathscr{U})$ (the set of all subsets of $\mathscr{U}$), ordered according to the set inclusion $\subseteq$. The meet and join of the power lattice is given by intersection and union. The bottom element is the empty set, that is $\perp = \emptyset$, and the top element is $\mathscr{U}$ itself, that is $\top = \mathscr{U}$.

Definition 2.3: Let P and Q be ordered sets. A map $f: P \to Q$ is said to be

(i) *order preserving* if $x \leq w \implies f(x) \leq f(w)$;
(ii) *order embedding* if $x \leq w \iff f(x) \leq f(w)$;
(iii) *order isomorphic* if it is order embedding and it maps P *onto* Q.

Every order isomorphic map faithfully mirrors the structure of P onto Q. In Figure 1 (down) we show some examples.

In the following section we examine how the properties of the functions f and h defining a transition system Σ are related to the observability properties of the system when the continuous variables are measured.

III. OBSERVABILITY CONDITIONS

Given a state transition system $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$, we give the following definitions.

Definition 3.1: The non empty sets $S_{(z_1,z_2)} = \{\alpha \in \mathscr{U} : z_2 = h(\alpha, z_1)\}$, for $z_1, z_2 \in \mathscr{Z}$, are named the *transition sets*. Each transition set contains all α values that allow the transition from z_1 to z_2 through the transition function h.

Definition 3.2: The set $\mathscr{Y} = \{Y_1, ..., Y_M\}$, with Y_i such that

(i) For any $Y_i \in \mathscr{Y}$ there is $(z_1, z_2) \in \mathscr{Z}$ such that $Y_i = S_{(z_1,z_2)}$;
(ii) For any $z_1, z_2 \in \mathscr{Z}$ for which $S_{(z_1,z_2)}$ is not empty, there is $j \in \{1, ..., M\}$ such that $S_{(z_1,z_2)} = Y_j$;

is the *set of transition classes*.

If the Y_i are not intersecting, the transition classes partition $\mathscr{U}$ in equivalence classes. We assume that all of the executions contained in the ω-limit set of Σ, $\omega(\Sigma)$, are distinguishable. More formally we have:

Assumption 3.1: The ω-limit set of $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$, $\omega(\Sigma)$, is such that for any two different executions σ_1, σ_2 with $\sigma_1(0), \sigma_2(0) \in \omega(\Sigma)$ there is $k \in \mathbb{N}$ such that $g(\sigma_1(k)) \neq g(\sigma_2(k))$.

Lemma 3.1: Consider the system $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$ and let the ω-limit set of Σ verify Assumption 3.1. Then Σ is observable if and only if for any $z_1, z_2 \in \mathscr{Z}$ we have that $f: (S_{(z_1,z_2)}, z_1) \to f(S_{(z_1,z_2)}, z_1)$ is one to one.

Proof: ($\Longrightarrow$). Let us show that if the system is observable then for any $z_1, z_2 \in \mathscr{Z}$ we have that $f: (S_{(z_1,z_2)}, z_1) \to f(S_{(z_1,z_2)}, z_1)$ is one to one. We have to show that if $\alpha_a \neq \alpha_b$ and $\alpha_a, \alpha_b \in S_{(z_1,z_2)}$ for some $z_1, z_2 \in \mathscr{Z}$, then $f(\alpha_a, z_1) \neq f(\alpha_b, z_1)$. Suppose instead that $f(\alpha_a, z_1) = f(\alpha_b, z_1)$, this means that the two executions σ_a, σ_b starting at $\sigma_a(0) = (\alpha_a, z_1)$ and $\sigma_b(0) = (\alpha_b, z_1)$ have the same output sequence, but they are different. This means that they are not distinguishable and therefore the system is not observable. This contradicts the assumption.

($\Longleftarrow$). We assume that for any $z_1, z_2 \in \mathscr{Z}$ we have that $f: (S_{(z_1,z_2)}, z_1) \to f(S_{(z_1,z_2)}, z_1)$ is one to one, and that Assumption 3.1 is verified. We need to show that for any $\sigma_1 \neq \sigma_2$ there is $k \in \mathbb{N}$ such that $g(\sigma_1(k)) \neq g(\sigma_2(k))$, that is σ_1 and σ_2 are distinguishable. Let then σ_1 and σ_2 be two executions such that $\sigma_1(0) \neq \sigma_2(0)$. Assume that $g(\sigma_1) = g(\sigma_2)$. This implies that $\sigma_1(k) = (\alpha_1(k), z(k))$ and $\sigma_2(k) = (\alpha_2(k), z(k))$. This implies that $\alpha_1(k) \neq \alpha_2(k)$ for all k, because $\alpha_1(k)$ and $\alpha_2(k)$ are in $S_{(z(k+1),z(k))} = \{\alpha \in \mathscr{U} : z(k+1) = h(\alpha, z(k))\}$, and we assumed that if $\alpha_1(k) \neq \alpha_2(k)$ then $f(\alpha_1(k), z(k)) \neq f(\alpha_2(k), z(k))$. This in turn implies that for $k \geq k_{\sigma_1}$ and $k \geq k_{\sigma_2}$, $\sigma_1(k) \in \omega(f, h)$, $\sigma_2(k) \in \omega(\Sigma)$, $\sigma_1(k) \neq \sigma_2(k)$ and $g(\sigma_1) = g(\sigma_2)$. This contradicts the assumption. Therefore if $\sigma_1(0) \neq \sigma_2(0)$, we have that $g(\sigma_1) \neq g(\sigma_2)$, which implies that σ_1 and σ_2 are distinguishable and by Definition 2.2 implies that Σ is observable with output z. ■

This Lemma shows that observability can be determined by checking if the transition function f is one to one on the transition sets $S_{(z_1,z_2)}$, provided that the executions evolving in the ω-limit set of Σ are distinguishable. This Lemma is used in the following theorem, which shows that if a system is observable there is always a lattice $(\chi, \leq)$ to which the maps f and h can be extended, so that the extensions $\tilde{f}$ and $\tilde{h}$ satisfy the conditions for the construction the LU discrete state estimator provided in [11].

Theorem 3.1: (Observability on bounded lattices) Consider the system $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$ and let the ω-limit set verify Assumption 3.1. Then the following are equivalent

1. The system Σ is observable;
2. There exist a bounded lattice $(\chi, \leq)$ with $\mathscr{U} \subseteq \chi$, with extensions $\tilde{h}: \chi \times \mathscr{Z} \to \mathscr{Z}$ such that $\tilde{h}|_{\mathscr{U} \times \mathscr{Z}} = h$, and $\tilde{f}: \chi \times \mathscr{Z} \to \chi$, such that $\tilde{f}|_{\mathscr{U} \times \mathscr{Z}} = f$ of h and f such that
   2a. $A_y(k) := \{x \in \chi : y(k+1) = \tilde{h}(y(k), x)\} = [l_y(k), u_y(k)]$, for some $l_y(k), u_y(k) \in \chi$, for any k;
   2b. $\tilde{f}: ([l_y(k), u_y(k)], y(k)) \to [\tilde{f}(l_y(k)), \tilde{f}(u_y(k))]$ is an order isomorphism for any k;

Proof: (1. $\Longrightarrow$ 2.) To show the existence of a lattice $(\chi, \leq)$ and extensions $\tilde{h}$ and $\tilde{f}$ that satisfy 2., we construct them. The proof proceeds in three steps. (0) We show that we can always construct a bounded lattice $(\chi, \leq)$ from a bounded set $\mathscr{U}$, with $\mathscr{U} \subset \chi$, by choosing a partial order on χ. (1) Given $(\chi, \leq)$ we show how to extend h to it to obtain a new map $\tilde{h}$ with the property 2a.. (2) Given $(\chi, \leq)$, given $\tilde{h}$, and given that Σ is observable, we use Lemma 3.1 to show how to extend f to $(\chi, \leq)$ to obtain a new map $\tilde{f}$ with property 2b..

(0) We define $(\chi, \leq) = (\mathscr{P}(\mathscr{U}), \subseteq)$. In this construction the elements in $\mathscr{U}$ are not related, therefore $\mathscr{U}$ is an antichain of $(\chi, \leq)$ (see Figure 2).

(1) Let α_i denote any element in $\mathscr{U}$, and let x denote an element in $\chi - \mathscr{U}$. The set of all pairs $(z_1, z_2) \in \mathscr{Z}$ for

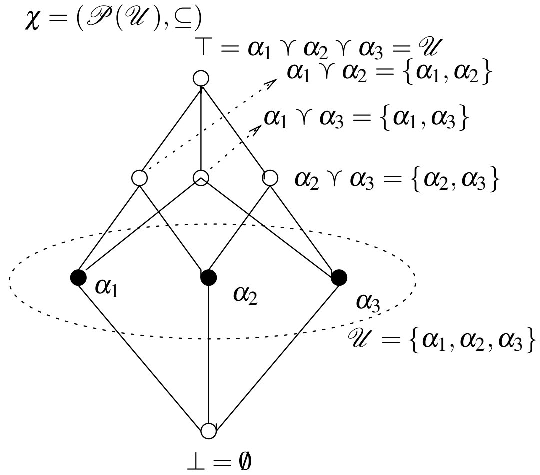

Fig. 2. Lattice $(\chi, \leq)$ in the case $\mathscr{U}$ is composed by three elements.

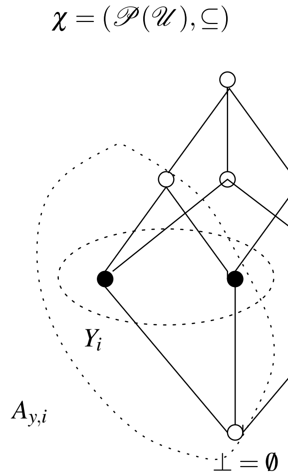

Fig. 3. Output lattices $A_{y,i}$.

which there is $\alpha \in \mathscr{U}$ such that $z_2 = h(\alpha, z_1)$ group the elements of $\mathscr{U}$ in the sets $S_{(z_1,z_2)} = \{\alpha \in \mathscr{U} : z_2 = h(\alpha, z_1)\}$. We add in each set elements of $\chi - \mathscr{U}$ in order to create a lattice interval in $(\chi, \leq)$ for each set. Thus if $S_{(z_1,z_2)} = \{\alpha : z_2 = h(\alpha, z_1)\} = \{\alpha_1, ..., \alpha_m\}$ we define $z_2 = \tilde{h}(x, z_1)$ for any $x = \alpha_{i_1} \curlyvee ... \curlyvee \alpha_{i_p}$ with $i_j \in \{1, ..., m\}$ and $p \leq m$. If instead $x = \alpha_{j_1} \curlyvee ... \curlyvee \alpha_{j_p}$ with α_{j_k} and α_{j_l}, for some $l, k \leq p$, such that $\alpha_{j_k} \in S_{(z_1,z_2)}$ and $\alpha_{j_l} \notin S_{(z_1,z_2)}$ for some z_1, z_2, we create a new set containing x only and different from all the others. This can be done for example, by taking a pair (z_1, z_2) and by defining $z_2 = \tilde{h}(x, z_1)$ for (z_1, z_2) such that there is no $w \in \chi$ such that $z_2 = \tilde{h}(w, z_1)$ unless $w = x$. We also define $z_2 = \tilde{h}(\perp, z_1)$ for any pair $(z_1, z_2) \in \mathscr{Z}$ for which $S_{(z_1,z_2)}$ is not empty. This is useful for constructing the output lattice intervals. In fact, for any $(z_1, z_2) \in \mathscr{Z}$ for which $S_{(z_1,z_2)}$ is not empty, we have that the set of all $x \in \chi$ for which $z_2 = \tilde{h}(x, z_1)$, is by construction a lattice of the form $[\perp, U]$, where $U = \alpha_{j_1} \curlyvee ... \curlyvee \alpha_{j_p}$, for some $j_1, j_2, ..., j_p$, $j_i \in \{1, ..., n\}$, $p \leq n$, and $S_{(z_1,z_2)} \subset [\perp, U]$ is an antichain of $[\perp, U]$. By construction it is clear that $\tilde{h}|_{\mathscr{U} \times \mathscr{Z}} = h$, and that $A_y(k) := \{x \in \chi : y(k+1) = \tilde{h}(y(k), x)\}$, with $y = z$, is by construction of the form $[l_y(k), u_y(k)]$, for some $l_y(k), u_y(k) \in \chi$. In particular, by our construction $l_y(k) = \perp$ for all k. Also by construction $A_y(k) \cap \mathscr{U} = Y_i$ for some $i \in \{1, ..., M\}$; let us define $A_{y,i} := A_y(k)$, for any k, if $A_y(k) \cap \mathscr{U} = Y_i$. Thus, each one of the transition classes Y_i as defined in 3.2 corresponds to one output lattice $A_{y,i}$, with $A_{y,i} = \mathscr{P}(Y_i)$ (see Figure 3).

(2) Once the elements of χ have been constructed and have been assigned an output value, we can define $\tilde{f}$ on all of the elements not in $\mathscr{U}$ in the following way. For simplifying notation we omit the dependence of f and $\tilde{f}$ on z, and we consider z as a parameter. First define $\tilde{f}(\perp) = \perp$. Then for any $x, w \in \chi$ define

$$\tilde{f}(x \curlyvee w) := f(x) \curlyvee f(w),$$
$$\tilde{f}(x \curlywedge w) := f(x) \curlywedge f(w) \qquad (1)$$

This implies that $\tilde{f} : \chi \longrightarrow \chi$ is order preserving. In fact, if $x < w$, by the way in which $(\chi, \leq)$ has been constructed, we have $x = \alpha_1 \curlyvee ... \curlyvee \alpha_p$ and $w = \alpha_1 \curlyvee ... \curlyvee \alpha_p \curlyvee .... \curlyvee \alpha_m$, for some $p \leq m$. Since $\tilde{f}(x) = f(\alpha_1) \curlyvee ... \curlyvee f(\alpha_p)$ and $\tilde{f}(w) = f(\alpha_1) \curlyvee ... f(\alpha_p) \curlyvee .... \curlyvee f(\alpha_m)$, we have that $\tilde{f}(w) = \tilde{f}(x) \curlyvee \tilde{f}(v)$ with $v = \alpha_{p+1} \curlyvee ... \curlyvee \alpha_m$. This implies that $\tilde{f}(x) \leq \tilde{f}(w)$. To show that $\tilde{f}$ is also an order embedding, by (ii) of Definition 2.3, we need to show also that if $\tilde{f}(x) \leq \tilde{f}(w)$ then $x \leq w$ for $x, w \in A_y(k)$. By assumption, the system is observable. This implies, by Lemma 3.1, that any $\alpha_i, \alpha_j \in S_{(z_1,z_2)}$ for $z_1, z_2 \in \mathscr{Z}$ are such that $f(\alpha_i, z_1) \neq f(\alpha_j, z_1)$. This along with relations (1) implies that if $x \neq w$ and $x, w \in A_y(k)$, with $S_{(z_1,z_2)} \subset A_y(k)$, we have that $\tilde{f}(x) \neq \tilde{f}(w)$. Therefore for any $x, w \in A_y(k)$, if $\tilde{f}(x) \leq \tilde{f}(w)$ we have that either (a) $f(x) < f(w)$ or (b) $f(x) = f(w)$. (a) implies by the order preserving property that $x < w$, (b) implies that $x = w$. Thus, $\tilde{f}(x) \leq \tilde{f}(w)$ implies $x \leq w$.

To show that $\tilde{f} : [l_y(k), u_y(k)] \to [\tilde{f}(l_y(k)), \tilde{f}(u_y(k))]$ is an order isomorphism, we need to show that it is also onto. In particular, we have to show that for any $w \in [\tilde{f}(l_y(k)), \tilde{f}(u_y(k))]$ there is a $x \in [l_y(k), u_y(k)]$ such that $w = \tilde{f}(x)$. By the definition of $(\chi, \leq)$ and A_y we have that $l_y(k) = \perp$. Also $u_y(k) = \alpha_1 \curlyvee ... \curlyvee \alpha_p$ for some $\alpha_i \in \mathscr{U}$. By relations (1) we have that $\tilde{f}(u_y(k)) = f(\alpha_1) \curlyvee ... \curlyvee f(\alpha_p)$. If $w \leq \tilde{f}(u_y(k))$, we also have that $w \leq f(\alpha_1) \curlyvee ... \curlyvee f(\alpha_p)$. This means, by the way the order is defined in (0), that $w = f(\alpha_{i_1}) \curlyvee ... \curlyvee f(\alpha_{i_m})$ for $i_j \in \{1, .., p\}$. By relations (1) it follows that $w = \tilde{f}(\alpha_{i_1} \curlyvee ... \curlyvee \alpha_{i_m})$ with $\alpha_{i_1} \curlyvee ... \curlyvee \alpha_{i_m} \leq \alpha_1 \curlyvee ... \curlyvee \alpha_p = u_y(k)$.

2. ($\Longrightarrow$ 1). To show that 2. implies that Σ is observable, we can apply Lemma 3.1. In particular, 2b. implies that $\tilde{f} : [l_y(k), u_y(k)] \to [\tilde{f}(l_y(k)), \tilde{f}(u_y(k))]$ is one to one, and therefore since $S_{(z_1,z_2)} \subset [l_y(k), u_y(k)]$ we also have that $f : S_{(z_1,z_2)} \longrightarrow f(S_{(z_1,z_2)})$ is one to one as well. This, along with Assumption 3.1, by Lemma 3.1 imply that the system is observable. ■

This Theorem states that if system Σ is observable with measured continuous variables, the result given in [11], which we report here for completeness, can be applied.

Theorem 3.2: (LU estimator) Consider the transition system $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$. If there exist a lattice $(\chi, \leq)$ with $\mathscr{U} \subseteq \chi$ such that

(i) The map $h : \mathscr{U} \times \mathscr{Z} \to \mathscr{Z}$ can be extended to $(\chi, \leq)$ as $\tilde{h} : \chi \times \mathscr{Z} \to \mathscr{Z}$, such that $\tilde{h}|_{\mathscr{U}\times\mathscr{Z}} = h$ and $A_y(k) := \{x \in \chi : y(k+1) = \tilde{h}(y(k), x)\} = [l_y(k), u_y(k)]$, for some $l_y(k), u_y(k) \in \chi$ for any k;

(ii) The map $f : \mathscr{U} \times \mathscr{Z} \to \mathscr{U}$ can be extended to $(\chi, \leq)$ as $\tilde{f} : \chi \times \mathscr{Z} \to \chi$, such that $\tilde{f}|_{\mathscr{U}\times\mathscr{Z}} = f$ and $\tilde{f} : (A_y(k), y(k)) \to [\tilde{f}(l_y(k), y(k)), \tilde{f}(u_y(k), y(k))]$ is an order isomorphism for any k;

(iii) System Σ is observable,

then the following system

$$L(k+1) = \tilde{f}(L(k) \curlyvee l_y(k), y(k)) \ , \quad (2)$$

$$U(k+1) = \tilde{f}(U(k) \curlywedge u_y(k), y(k)) \ , \quad (3)$$

with $L(0) = \bigwedge \chi$ and $U(0) = \bigvee \chi$, is such that

(a) $\alpha(k) \in [L(k), U(k)]$ for all k (correctness);

(b) $|[L(k+1), U(k+1)]| \leq |[L(k), U(k)]|$ (non-increasing error);

(c) $|[L(k), U(k)] \cap \mathscr{U} - \alpha| \to 0$ as $k \to \infty$ (convergence).

Moreover, if the extended system $\tilde{\Sigma} = (\tilde{f}, \tilde{h}, \chi, \mathscr{Z})$ defined on $\chi \times \mathscr{Z}$ with output z is also observable, property (c) becomes:

(c') $L(k) \to \alpha(k)$ and $U(k) \to \alpha(k)$ as $k \to \infty$.

We refer to the estimator in equations (2) and (3) as LU estimator, where $L(k)$ and $U(k)$ are the lower and upper bounds respectively, according to the chosen order $(\chi, \leq)$, of the set of all possible α values that are compatible with the output sequence and with the dynamics of the system Σ.

IV. COMPLEXITY CONSIDERATIONS

The complexity of the estimator given by expressions (2) and (3) is due to the complexity needed for computing $\tilde{f}$ on elements of $(\chi, \leq)$ as well as the one needed for computing joins and meets between elements in $(\chi, \leq)$. If for computing them, we need to store all of the elements of $(\chi, \leq)$ along with their order relations, we incur in serious computational issues when $(\chi, \leq)$ is large. The order relations among elements in $(\chi, \leq)$ can be efficiently established on-line when $(\chi, \leq)$ is equipped with algebraic properties. In this case, if $\tilde{f}$ can be computed efficiently using the same algebraic properties, the size of $(\chi, \leq)$ is not a cause of complexity.

In this section, we focus on the case in which $(\chi, \leq)$ needs to be stored, and we compute its worst-case size when the discrete state dynamics is determined by an automaton so to have a term of comparison with previous results. In this case, $f : \mathscr{U} \to \mathscr{U}$ does not depend on z. The following proposition shows that the size of $(\chi, \leq)$ is at most $2N^2$ where $N = |\mathscr{U}|$. This shows that the worst case computation needed for implementing our estimator is the same as the one needed in Caines [4], where the observer tree method is proposed. The main advantage of this method is clear when the space of discrete variables can be immersed in a lattice whose order relations can be computed algebraically ($(\chi, \leq)$ does not need to be stored), as it happens in the multi-robot example proposed in [11].

Proposition 4.1: Consider the system $\Sigma = (f, h, \mathscr{U}, \mathscr{Z})$, with $f : \mathscr{U} \to \mathscr{U}$, and let $N = |\mathscr{U}|$. Assume that the sets $\{Y_1, ..., Y_m\}$ are all disjoint. Then,

$$|\chi| \leq 2N^2.$$

Proof: We construct the worst case $(\chi, \leq)$ by adding in it all the elements that the estimator in Theorem 3.2 needs. These are in the set of subsets of $\mathscr{U}$ ordered according to inclusion. The proof proceeds in two steps: 1) we show that for any Y_i, the last element of the sequence $\{Y_i, f(Y_i) \cap Y_{i_1}, f(f(Y_i) \cap Y_{i_1}) \cap Y_{i_2}, ..., f(f(...f(Y_i) \cap Y_{i_1}...)) \cap Y_{i_n}\}$ is a singleton for $n < N$, for any $i_j \in \{1, ..., m\}$; 2) the *jth* element of the above sequence can generate at most N nonempty sets for any combination $(i_1, ..., i_{j-1})$ and for any j.

Proof of 1). Let ω_α denote the ω-limit set of f. Since all of the executions are converging to the ω-limit set, it is enough to show that any two execution starting in the ω_α, will distinguish from each other in less than $n = |\omega_\alpha|$. We proceed by contradiction. Assume that there are $x_i, x_j \in \omega_\alpha$ such that

$$(a)\ Y(f^k(x_i)) = Y(f^k(x_j)) \text{ for any } k < n \text{ and}$$

$$(b)\ Y(f^n(x_i)) \neq Y(f^n(x_j)),$$

where $Y(x) = Y_j$ if $x \in Y_j$. Since x_i and x_j belong each to a limit cycle, there are k_j, k_i such that $f^{n-k_i}(x_i) = f^n(x_i)$ and $f^{n-k_j}(x_j) = f^n(x_j)$. As a consequence, we have by (b) that

$$(c)\ Y(f^{n-k_i}(x_i)) \neq Y(f^{n-k_j}(x_j)).$$

Assume without loss in generality that $k_i \geq k_j$. If x_i and x_j belong to the same limit cycle, we have $k_i = k_j$, and therefore we contradict (a). If $k_i > k_j$, x_i and x_j belong to different limit cycles, and k_i and k_j are the respective limit cycle lengths. Thus $k_i + k_j \leq n$. Thus, by virtue of (a) we have $Y(f^{n-(k_i+k_j)}(x_i)) = Y(f^{n-(k_i+k_j)}(x_j))$ and by the periodicity of trajectories in the limit cycles, we have $f^{n-(k_i+k_j)}(x_i) = f^{n-k_j}(x_i)$ and $f^{n-(k_i+k_j)}(x_j) = f^{n-k_i}(x_j)$. As a consequence, $Y(f^{n-(k_i+k_j)}(x_i)) = Y(f^{n-k_j}(x_i))$ and $Y(f^{n-(k_i+k_j)}(x_j)) = Y(f^{n-k_i}(x_j))$. By (a), we also have that $Y(f^{n-k_i}(x_j)) = Y(f^{n-k_i}(x_i))$ and $Y(f^{n-k_j}(x_i)) = Y(f^{n-k_j}(x_j))$. One can verify that the set of these relations are inconsistent with (c).

Proof of 2). Since the Y_is are all disjoint, the *jth* element of the sequence in 2) can have at most $|Y_i|$ nonempty intersections for any combination of $(i_1, ..., i_{j-1})$ for any j. Then for any j, we can have at most $\sum_i |Y_i| = N$ nonempty intersections.

Since the estimator needs all of these N^2 elements and the "f" of these elements, the size of $(\chi, \leq)$ is at most $2N^2$. ■

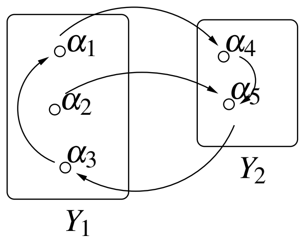

Fig. 4. Automaton example.

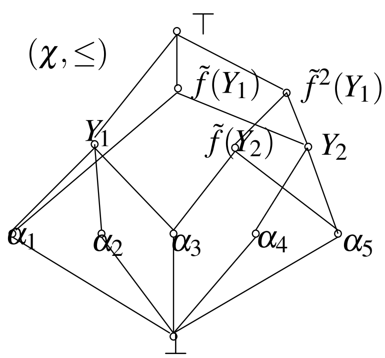

Fig. 5. Automaton example: lattice $(\chi,\leq)$.

A. Examples

In the first example, we show how the lattice $(\chi,\leq)$ looks like in the case of an automaton driving the discrete state dynamics. Consider the automaton reported in Figure 4 where $\mathscr{U}=\{\alpha_1,..,\alpha_5\}$, and $\mathscr{Y}=\{Y_1,Y_2\}$. The lattice $(\chi,\leq)$ can be constructed by following the procedure in the proof of Proposition 4.1, and it is shown in Figure 5.

Next, we consider the multi-robot example proposed in [11]. In the case of N robots, the worst case size predicted by Proposition 4.1 is $(N!)^2$. This is larger than the size of the lattice used for the construction of the estimator, which is N^N, as $(\chi,\leq)$ is chosen to be the set of vectors with natural entries between 1 and N. However, the complexity reduction obtained by the above choice of the lattice is due to its algebraic properties and not to its size, which is still large for large number of agents N.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper we have considered the observability properties of a class of hybrid systems where the continuous variables are measured. We have shown that if the system is observable, there always exist a lattice on which the system can be extended and the LU estimator can be constructed. Such an estimator updates the least and greatest elements of the set of all possible discrete variable values compatible with the output sequence. This shows that the lattice approach to the estimation problem is general. In particular, we have shown that in the case in which the discrete variable evolution is determined by an automaton, the estimator we propose is analogous in terms of computational burden to the one proposed in Caines [4]. The advantage of using the lattice approach for state estimation appears clear when the space of discrete variables can be immersed in a lattice whose order relation can be computed algebraically. This is the case of the example proposed in [11], where the estimation problem is prohibitive if the partial order is not explicitly exploited in the estimator design.

B. Future Works

One main research trust for the future is to develop an estimator design that allows to estimate the continuous variables as well. The authors expect that under suitable order preserving properties satisfied by the continuous and discrete state evolution, part of the computational burden associated with the state estimation problem will be overcome.

VI. ACKNOWLEDGMENTS

This work has been funded in part by the ONR grant N00014-10-1-0890 under the MURI program.

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