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Cascade estimators for systems on a partial order

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ABSTRACT

In this work, the problem of estimating the state in systems with continuous and discrete variables is considered. A cascade state estimator on a partial order is constructed and conditions for its existence are provided. This work has two main contributions. First, it extends existing state estimation algorithms on a partial order to estimate also the continuous variables. Second, it shows that the proposed construction is general.

1. Introduction

Hybrid system models have become increasingly popular as a framework for describing discrete and continuous state dynamics that characterize embedded systems. The problem of state estimation arises as a means for control under partial observation and as a means for fault diagnosis. Estimating the values of non-measurable variables in hybrid systems with reasonable computational effort is challenging. In the worst case, the size of the set of possible current discrete states can grow exponentially with the number of measurements due to the coupling of continuous and discrete dynamics. We address this computational challenge by proposing an alternative approach to enumeration techniques, which exploits a partial order structure on the set of continuous and discrete states. There is a wealth of research on the problem of estimating the state of hybrid systems. Bemporad et al. [3] propose a deadbeat observer for piecewise affine systems, which requires large amounts of computation. Balluchi et al. [2] combine a *location* observer with a Luenberger observer. However, if the number of locations is large, as in the systems that we consider, such an approach is impracticable. In Alessandri et al., Luenberger-like observers are proposed for hybrid systems, but the system location is known [1]. Özveren et al. [7] and Caines [4] propose discrete event observers based on the construction of the current-location observation tree, which is impractical when the number of locations is large. The main contribution of this work is a new approach to state estimation that exploits partial order structures. The idea is the one of finding an alternative way to enumeration in order to *represent* the sets of interest. This alternative way relies on representing sets by means of lower and upper bounds in suitable partial orders. This approach was first proposed in the author's previous work [10], in which only the discrete variables were estimated, while the continuous variables were available for measurement. In the survey paper [9], the results on state estimation on partial orders are summarized. In this paper, this approach is extended to the case in which the continuous variables also need to be estimated. We show that the proposed approach is general as partial orders on which to construct the estimator can always be found provided that the system has observability properties. The computational load of the estimator is highly dependent on the specific partial order chosen. We thus show through examples what classes of systems allow for partial order choices that lead to low computation estimators. In Section 2, we introduce a multi-robot example to explain the basic idea. In Section 3, we introduce basic notions on partial orders and the system model. Section 4 formulates the state estimation problem and gives a solution. In Section 5, the existence of the estimator is investigated. In Section 6, we illustrate several examples.

2. A multi-robot example

As an illustrative example, we consider a task that represents a defensive maneuver for a robotic "capture the flag" game [5]. In this example, as opposed to [10], the continuous variables are only partially measured. Some number of blue robots with positions $(z_{i,1}, 0) \in \mathbb{R}^2$ (denoted by open circles) and with speeds $(z_{i,2}, 0) \in \mathbb{R}^2$ must defend their zone $\{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}$ from an equal number of incoming red robots (denoted by filled circles). The

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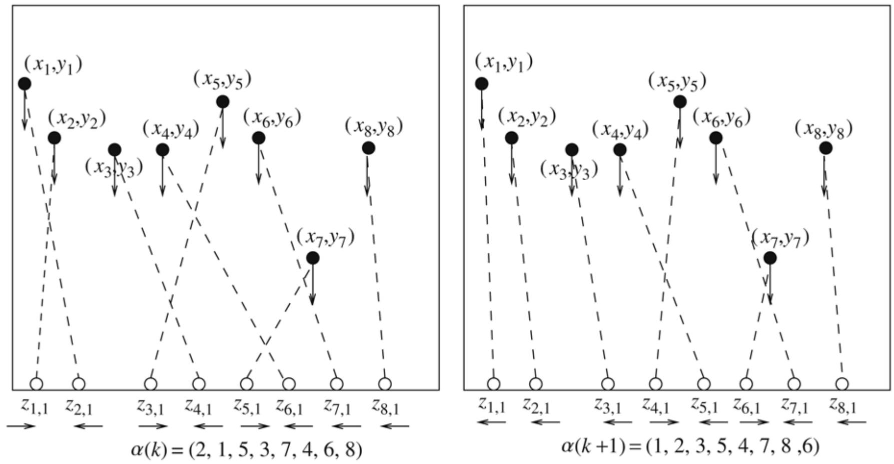

Fig. 1. Example of the RoboFlag Drill with 8 robots per team. The dashed lines represent the assignment of each blue robot to red robot. The arrows denote the direction of motion of each robot.

positions of the red robots are $(x_i, y_i) \in \mathbb{R}^2$ (Fig. 1). The red robots move straight toward the blue robots' defensive zone. The blue robots are each assigned to a red robot, and they coordinate to intercept the red robots. Let N represent the number of robots in each team. The robots start with an arbitrary (bijective) assignment $\alpha : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\}$, where α_i is the red robot that blue robot i is required to intercept. At each step, each blue robot communicates with its neighbors and decides to either switch assignments with its left or right neighbor or keep its assignment. The RoboFlag Drill system can be specified by the following rules: $y_i(k+1) = y_i(k) - \delta$ if $y_i(k) \geq \delta$ and

$$z'_{i,1} = (1-\beta)z_{i,1} - \beta z_{i,2} + 2\beta x_{\alpha_i} \tag{1}$$

$$z'_{i,2} = (1-\lambda)z_{i,2} + \lambda x_{\alpha_i} \tag{2}$$

$$(\alpha_i(k+1), \alpha_{i+1}(k+1)) = (\alpha_{i+1}(k), \alpha_i(k))$$
$$\text{if } x_{\alpha_i(k)} \geq z_{i+1,1}(k) \wedge x_{\alpha_{i+1}(k)} \leq z_{i+1,1}(k), \tag{3}$$

where we assume that $x_i < z_{i,1}(k) < x_{i+1}$ and $x_i < z_{i,2}(k) < x_{i+1}$ for all i and all k, which is guaranteed if β and λ are sufficiently small. This implies that each defender moves toward the x position of the assigned attacker with second order damped dynamics. Eq. (3) establishes that two robots trade their assignments if the current assignments cause them to go toward each other. Given the evolution of the measurable quantities $z_{i,1}$, x_i, and y_i for all i, can we build an estimator that tracks on-line the value of the assignment $\alpha(k)$ and of the robots speeds $z_{i,2}(k)$ for all k? The value of $\alpha \in \text{perm}(N)$ determines the discrete state. The number of possible discrete states is $N!$. This renders prohibitive the application of observers based on the current-location observation tree [4,2,7]. Consider the situation depicted in Fig. 1(left) where $N = 8$. We see the blue robots 1, 3, 5 going right and the others going left. From Eq. (1)–(2) with $x_i < z_{i,1} < x_{i+1}$ and $x_i < z_{i,2} < x_{i+1}$ we deduce that the set of all possible $\alpha \in \text{perm}(N)$ compatible with this observation is such that $\alpha_i \geq i+1$ for $i \in \{1, 3, 5\}$ and $\alpha_i \leq i$ for $i \in \{2, 4, 6, 7, 8\}$. According to enumeration methods, this set needs to be mapped forward through the dynamics of the system. Such a set is then intersected with the set of α values compatible with the new observation. For each possible discrete state, a continuous state estimator must be run to estimate the continuous variable corresponding to the discrete state. To overcome the complexity issue that comes from the need of listing order of $N!$ elements, we propose to represent a set by a lower L and an upper U elements according to some partial order. Then, we can perform the previously described operations only on L and U, two elements instead of $N!$. For this example, we can view $\alpha \in \mathbb{N}^N$. The set of possible assignments compatible with the observation of the z motion deduced from Eqs. (1) and (2), denoted $O_y(k)$, can be represented as the interval $[(2, 1, 4, 1, 6, 1, 1, 1), (8, 2, 8, 4, 8, 6, 7, 8)]$ with the order established component-wise. The function $\tilde{f}$ that maps such a set forward, specified by Eq. (3) with the assumption that $x_i < z_{i,1} < x_{i+1}$, simply swaps two adjacent robot assignments if these cause the two robots to move toward each other. Thus, it maps the set $O_y(k)$ to the set $\tilde{f}(O_y(k)) = [(1, 2, 1, 4, 1, 6, 1, 1), (2, 8, 4, 8, 6, 8, 7, 8)]$. When the new output measurement becomes available (Fig. 1(right)) we obtain the new set $O_y(k+1) = [(1, 1, 1, 5, 1, 7, 1, 1), (1, 2, 3, 8, 5, 8, 7, 8)]$. The sets $\tilde{f}(O_y(k))$ and $O_y(k+1)$ can be intersected by simply computing the supremum of their lower bounds and the infimum of their upper bounds to obtain $[(1, 2, 1, 5, 1, 7, 1, 1), (1, 2, 3, 8, 5, 8, 7, 8)]$. This way, we obtain the system that updates L and U, L and U being the lower and upper bounds of the set of all possible α compatible with the output sequence: $L(k+1) = \tilde{f}(\sup(L(k), \inf O_y(k)))$, $U(k+1) = \tilde{f}(\inf(U(k), \sup O_y(k)))$. The computational burden of this implementation is of the order of N. This computational burden is to be compared to $N!$, which is the computation requirement that we have with the enumeration approach. Once we have an interval in which the discrete state exists, we can determine the interval in which the continuous state $z_2 = (z_{1,2}, \ldots, z_{N,2})$ exists. Let $z_1 = (z_{1,1}, \ldots, z_{N,1})$ be the vector of positions. For a pair of consecutive measurements $z_1(k), z_1(k+1)$ and for $\alpha(k) \in [L^*(k), U^*(k)]$ with $L^*(k) = \sup(L(k), \inf O_y(k))$ and $U^*(k) = \inf(U(k), \sup O_y(k))$, we have that z_2 is also in an interval (using component-wise ordering), which is *induced* by the interval $[L^*(k), U^*(k)]$. We denote such an interval induced by $[L^*(k), U^*(k)]$ as $I^{[L^*(k),U^*(k)]}_{z_1(k),z_1(k+1)}$. This induced interval represents the set of all possible continuous variables z_2 that are compatible with observations $z_1(k)$ and $z_1(k+1)$ and with a discrete state in the interval $[L^*(k), U^*(k)]$. The ends of this induced interval can be easily computed by Eq. (1). In fact, the map that attaches to a value $\alpha \in \mathbb{N}^N$ the values $z_{i,2}$ for a pair of consecutive observations $z_1(k), z_1(k+1)$ is given for all i by $\alpha_i \rightarrow \frac{1}{\beta}((1-\beta)z_{1,i}(k) - z_{1,i}(k+1) + 2\beta x_{\alpha_i})$. If we denote by $M_{z_1(k),z_1(k+1)}(\alpha)$ the map that attaches to α, the value of z_2 for a given pair of consecutive observations $z_1(k), z_1(k+1)$, we obtain $M_{z_1(k),z_1(k+1)}(\alpha) = \frac{1}{\beta}(((1-\beta)z_{1,1}(k) - z_{1,1}(k+1) + 2\beta x_{\alpha_1}), \ldots, ((1-\beta)z_{N,1}(k) - z_{N,1}(k+1) + 2\beta x_{\alpha_N}))$. The ends of the induced interval are thus given by $\inf I^{[L^*(k),U^*(k)]}_{z_1(k),z_1(k+1)} = M_{z_1(k),z_1(k+1)}(L^*(k))$ and $\sup I^{[L^*(k),U^*(k)]}_{z_1(k),z_1(k+1)} = M_{z_1(k),z_1(k+1)}(U^*(k))$. The lower and upper bounds of the set of possible z_2 values, which we call z_L and z_U, can be updated by:

$$z_L(k+1) = \tilde{h}_2\left(\sup(z_L(k), \inf I^{[L^*(k),U^*(k)]}_{z_1(k),z_1(k+1)}), L^*(k)\right),$$
$$z_U(k+1) = \tilde{h}_2\left(\inf(z_U(k), \sup I^{[L^*(k),U^*(k)]}_{z_1(k),z_1(k+1)}), U^*(k)\right), \quad (4)$$

in which $\tilde{h}_2$ is the function that updates the variables z_2 as given in Eq. (2). In this paper, this construction is made general by employing partial order theory.

3. Basic notions

A partial order [6] is a set χ with a partial order relation "$\le$", and it is denoted $(\chi, \le)$. The *join* "$\curlyvee$" and the *meet* "$\curlywedge$" of two elements x and w in χ are defined as $x \curlyvee w = \sup\{x, w\}$ and $x \curlywedge w = \inf\{x, w\}$, if $S \subseteq \chi$, $\bigvee S = \sup S$ and $\bigwedge S = \inf S$, where $\sup\{x, w\}$ denotes the smallest element in χ that is larger than both x and w, and $\inf\{x, w\}$ denotes the largest element in χ that is smaller than both x and w. If $x < w$ and there is no other element in between x and w, we write $x \ll w$. Let $(\chi, \le)$ be a partial order. If $x \curlywedge w \in \chi$ and $x \curlyvee w \in \chi$ for all $x, w \in \chi$, then $(\chi, \le)$ is a *lattice*. Let $(\chi, \le)$ be a lattice and let $S \subseteq \chi$ be a non-empty subset of χ. Then $(S, \le)$ is a *sublattice* of χ if $a, b \in S$ implies that $a \curlyvee b \in S$ and $a \curlywedge b \in S$. If all sublattices of χ contain their least and greatest elements, then $(\chi, \le)$ is called *complete*. Given a complete lattice $(\chi, \le)$, we are concerned with a special kind of a sublattice called an *interval sublattice* defined as follows. Any interval sublattice of $(\chi, \le)$ is given by $[L, U] = \{w \in \chi \mid L \le w \le U\}$ for $L, U \in \chi$. That is, this special sublattice can be represented by only two elements. The cardinality of an interval sublattice $[L, U]$ is denoted $|[L, U]|$. The *power lattice* of a set $\mathcal{U}$, denoted $(\mathcal{P}(\mathcal{U}), \subseteq)$, is given by the power set of $\mathcal{U}$, $\mathcal{P}(\mathcal{U})$ (the set of all subsets of $\mathcal{U}$), ordered according to the set inclusion $\subseteq$. The meet and join of the power lattice are given by intersection and union. The bottom element is the empty set, that is $\bot = \emptyset$, and the top element is $\mathcal{U}$ itself, that is $\top = \mathcal{U}$. Let $(P, \le)$ and $(Q, \le)$ be partially ordered sets. A map $f : P \to Q$ is (i) an *order preserving map* if $x \le w \Rightarrow f(x) \le f(w)$; (ii) an *order embedding* map if $x \le w \iff f(x) \le f(w)$; (iii) an *order isomorphism* if it is order embedding and it maps P *onto* Q.

Definition 3.1 (*Distance on a Partial Order*). Let $(P, \le)$ be a partial order. A distance d on $(P, \le)$ is a function $d : P \times P \to \mathbb{R}$ such that the following properties are satisfied: (i) $d(x, y) \ge 0$ for all $x, y \in P$ and $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$; (iii) if $x \le y \le z$ then $d(x, y) \le d(x, z)$; (iv) $d(x, z) \le d(x, y) + d(y, z)$ (triangular inequality).

Note that any function d that satisfies the items in Definition 3.1 is a distance function. So, the distance function on a partial order is not unique. Let $(P_1, \le)$ and $(P_2, \le)$ be two partial orders. Their Cartesian product is given by $(P_1 \times P_2, \le)$, where $P_1 \times P_2 = \{(x, y) \mid x \in P_1 \text{ and } y \in P_2\}$ and $(x, y) \le (x', y')$ if and only if $x \le x'$ and $y \le y'$. For all $(p_1, p_2) \in P_1 \times P_2$ the standard projections $\pi_1 : P_1 \times P_2 \to P_1$ and $\pi_2 : P_1 \times P_2 \to P_2$ are such that $\pi_1(p_1, p_2) = p_1$ and $\pi_2(p_1, p_2) = p_2$. Let P_1 and P_2 be two sets with $P_1 \subseteq P_2$ and $(P_2, \le)$ a partial order. For all $x \in P_2$, we define the *lower and upper approximations* of x in P_1 as $a_L(x) := \max_{(P_2,\le)}\{w \in P_1 \mid w \le x\}$ and $a_U(x) := \min_{(P_2,\le)}\{w \in P_1 \mid w \ge x\}$. If such lower and upper approximations exist for all $x \in P_2$, then the partial order $(P_2, \le)$ is said to be *closed with respect to* P_1. One can verify that the lower and upper approximation functions are order preserving.

A *deterministic transition system* (DTS) is a tuple $\Sigma = (S, \mathcal{Y}, F, g)$, where (i) S is a set of states with $s \in S$; (ii) $\mathcal{Y}$ is a set of outputs with $y \in \mathcal{Y}$; (iii) $F : S \to S$ is the state transition function; (iv) $g : S \to \mathcal{Y}$ is the output function. An execution of a deterministic transition system Σ is all sequence $\sigma = \{s(k)\}_{k\in\mathbb{N}}$ such that $s(0) \in S$ and $s(k+1) = F(s(k))$ for all $k \in \mathbb{N}$. The set of all executions of Σ is denoted $\mathcal{E}(\Sigma)$. An output sequence of the system Σ corresponding to an execution σ is denoted $\{y(k)\}_{k\in\mathbb{N}}$ and it is such that $y(k) = g(s(k))$. The deterministic transition system $\Sigma = (S, \mathcal{Y}, F, g)$ is said to be *observable* if any two different executions $\sigma_1, \sigma_2 \in \mathcal{E}(\Sigma)$ are such that there exists a k such that $g(\sigma_1(k)) \ne g(\sigma_2(k))$. It is useful to define also systems with inputs and their interconnections. We define two types of interconnection: feedback interconnection and cascade interconnection. A *deterministic transition system with input* is the tuple $\Sigma = (S, \mathcal{I}, \mathcal{Y}, F, g)$, where (i) S is a set of states with $s \in S$; (ii) $\mathcal{I}$ is a set of inputs; (iii) $\mathcal{Y}$ is a set of outputs with $y \in \mathcal{Y}$; (iv) $F : S \times \mathcal{I} \to S$ is the state transition function; (v) $g : S \times \mathcal{I} \to \mathcal{Y}$ is the output function. Consider the two systems with inputs $\Sigma_1 = (\mathcal{S}_1, \mathcal{I}_1, \mathcal{Y}_1, F_1, g_1)$ and $\Sigma_2 = (\mathcal{S}_2, \mathcal{I}_2, \mathcal{Y}_2, F_2, g_2)$, in which $\mathcal{I}_1 = \mathcal{Y}_2$, $\mathcal{I}_2 = \mathcal{Y}_1$, and $g_1 : \mathcal{S}_1 \to \mathcal{Y}_1$. The *feedback interconnection* of Σ_1 with Σ_2, denoted by $\Sigma_1 \circ_f \Sigma_2$, is the deterministic transition system given by $\Sigma_1 \circ_f \Sigma_2 := (\mathcal{S}_1 \times \mathcal{S}_2, \mathcal{Y}_2, (F_1', F_2'), g_2')$, in which for all $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$, we have $F_1'(s_1, s_2) = F_1(s_1, g_2(s_2, g_1(s_1)))$, $F_2'(s_1, s_2) = F_2(s_2, g_1(s_1))$, and $g_2'(s_1, s_2) = g_2(s_2, g_1(s_1))$. The output of the feedback interconnection $\Sigma_1 \circ_f \Sigma_2$ is the output of Σ_2. Consider the two systems with inputs $\Sigma_1 = (\mathcal{S}_1, \mathcal{I}_1, \mathcal{Y}_1, F_1, g_1)$ and $\Sigma_2 = (\mathcal{S}_2, \mathcal{I}_2, \mathcal{Y}_2, F_2, g_2)$, in which $\mathcal{I}_2 = \mathcal{Y}_1$. The *cascade interconnection* of Σ_1 and Σ_2, denoted $\Sigma_1 \circ_c \Sigma_2$, is the deterministic system with input given by $\Sigma_1 \circ_c \Sigma_2 := (\mathcal{S}_1 \times \mathcal{S}_2, \mathcal{I}_1, \mathcal{Y}_2, (F_1', F_2'), g_2')$, in which for all $s_1 \in \mathcal{S}_1$, $s_2 \in \mathcal{S}_2$, and $u_1 \in \mathcal{I}_1$ we have that $F_1'(s_1, s_2, u_1) = F_1(s_1, u_1)$, $F_2'(s_1, s_2, u_1) = F_2(s_2, g_1(s_1, u_1))$, and $g_2'(s_1, s_2, u_1) = g_2(s_2, g_1(s_1, u_1))$. Let $\mathcal{U}$ be a finite discrete set, $\mathcal{Z}$ an infinite possibly dense set, and $\mathcal{Y}$ a finite or infinite set. In this paper, we consider systems of the form $\Sigma = \Sigma_1 \circ_f \Sigma_2$, in which $\Sigma_1 = (\mathcal{U}, \mathcal{Y}, \mathcal{U}, f, \text{id})$ and $\Sigma_2 = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, h, g)$, with $g : \mathcal{Z} \times \mathcal{U} \to \mathcal{Y}$, $f : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$, $h : \mathcal{Z} \times \mathcal{U} \to \mathcal{Z}$. Then, we have that $\Sigma_1 \circ_f \Sigma_2 = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f', h'), g')$, in which for all $\alpha, z \in \mathcal{U} \times \mathcal{Z}$ we have that $f'(\alpha, z) = f(\alpha, g(z, \alpha))$, $h'(\alpha, z) = h(z, \alpha)$, and $g'(\alpha, z) = g(z, \alpha)$. We attach to $\Sigma = \Sigma_1 \circ_f \Sigma_2$, the following difference equations $\alpha(k+1) = f(\alpha(k), y(k))$, $z(k+1) = h(z(k), \alpha(k))$, $y(k) = g(z(k), \alpha(k))$. In the following, we denote by $\sigma(z)(k)$ and by $\sigma(\alpha)(k)$ the values of the variables z and α along the execution σ, respectively. We next define the set of all possible discrete variable values that are compatible with two consecutive output measurements. The sets $T_{y_1,y_2}(\Sigma) = \{\alpha \in \mathcal{U} \mid \exists\, z \in \mathcal{Z} \text{ such that } y_1 = g(z, \alpha) \text{ and } y_2 = g(h(z, \alpha), f(\alpha, y_1))\}$ with $y_1, y_2 \in \mathcal{Y}$ are the *Σ-transition sets*. We denote the property of the system Σ that allows us to distinguish two different initial values of the variables α independently of the continuous state by independent discrete state observability. The system Σ is said to be *independently discrete state observable* if for all output sequences $\{y(k)\}_{k\in\mathbb{N}}$, we have that for any two executions $\sigma_1, \sigma_2 \in \mathcal{E}(\Sigma)$ such that $\{\sigma_1(k)(\alpha)\}_{k\in\mathbb{N}} \ne \{\sigma_2(k)(\alpha)\}_{k\in\mathbb{N}}$, there is $k > 0$ such that $\sigma_1(k)(\alpha) \in T_{y(k),y(k+1)}(\Sigma)$ and $\sigma_2(k)(\alpha) \notin T_{y(k),y(k+1)}(\Sigma)$. This property basically states that an independently discrete state observable system is such that any two executions with different discrete state sequences cannot have the same output sequence, that is, at some point $\sigma_1(k)(\alpha)$ is compatible with the output pair $y(k), y(k+1)$ ($\sigma_1(k)(\alpha) \in T_{y(k),y(k+1)}(\Sigma)$), but $\sigma_2(k)(\alpha)$ is not ($\sigma_2(k)(\alpha) \notin T_{y(k),y(k+1)}(\Sigma)$). This property allows us to construct a discrete-continuous state estimator that is a cascade interconnection of a discrete state estimator as the one in [10], and a continuous state estimator.

4. Problem statement and solution

Consider the deterministic transition system $\Sigma = \Sigma_1 \circ_f \Sigma_2$, with output sequence $\{y(k)\}_{k\in\mathbb{N}}$. From the measurement of the output sequence, we want to construct a cascade state estimator: A system $\hat{\Sigma} = \hat{\Sigma}_1 \circ_c \hat{\Sigma}_2$, in which $\hat{\Sigma}_1$ takes as input the values of the output of Σ and asymptotically tracks the value of the variables α, while $\hat{\Sigma}_2$ takes as input the discrete state estimates and asymptotically tracks the value of z.

Problem 1 (*Cascade State Estimator*). Given the deterministic transition system $\Sigma = \Sigma_1 \circ_f \Sigma_2$, in which $\Sigma_1 = (\mathcal{U}, \mathcal{Y}, \mathcal{U}, f, \text{id})$ and $\Sigma_2 = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, h, g)$, determine the cascade interconnection $\hat{\Sigma} = \hat{\Sigma}_1 \circ_c \hat{\Sigma}_2$, in which $\hat{\Sigma}_1 = (\chi \times \chi, \mathcal{Y} \times \mathcal{Y}, \chi \times \chi, (f_1, f_2), \text{id})$ with $f_1 : \chi \times \mathcal{Y} \times \mathcal{Y} \to \chi, f_2 : \chi \times \mathcal{Y} \times \mathcal{Y} \to \chi$, $\hat{\Sigma}_2 = (\mathcal{L} \times \mathcal{L}, \chi \times \chi \times \mathcal{Y} \times \mathcal{Y}, \chi \times \chi \mathcal{Z}_E \times \mathcal{Z}_E, (f_3, f_4), (g_1, g_2))$ with $f_3 : \mathcal{L} \times \chi \times \chi \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{L}$, $f_4 : \mathcal{L} \times \chi \times \chi \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{L}, g_1 : \chi \to \chi, g_2 : \chi \to \chi$, with $g_1 = g_2 = \text{id}, g_3 : \mathcal{L} \to \mathcal{Z}_E$, and $g_4 : \mathcal{L} \to \mathcal{Z}_E$, with $\mathcal{U} \subseteq \chi$, $(\chi, \leq)$ a lattice, $\mathcal{Z} \subseteq \mathcal{Z}_E$ with $(\mathcal{Z}_E, \leq)$ a lattice, $\chi \times \mathcal{Z}_E \subseteq \mathcal{L}$ with $(\mathcal{L}, \leq)$ a lattice, such that for all executions $\sigma = \{(\alpha(k), z(k))\}_{k\in\mathbb{N}}$ of Σ with output sequence $\{y(k)\}_{k\in\mathbb{N}}$ the update laws

$$\begin{aligned} L(k+1) &= f_1(L(k), y(k), y(k+1)), \\ U(k+1) &= f_2(U(k), y(k), y(k+1)), \\ q_L(k+1) &= f_3(q_L(k), L(k), U(k), y(k), y(k+1)), \\ q_U(k+1) &= f_4(q_U(k), L(k), U(k), y(k), y(k+1)), \end{aligned} \tag{5}$$

with $z_L(k) = g_3(q_L(k))$, and $z_U(k) = g_4(q_U(k))$, in which $L(0) := \bigwedge \chi$, $U(0) := \bigvee \chi$, $q_L(0) = \bigwedge \mathcal{L}$, $q_U(0) = \bigvee \mathcal{L}$, have the following properties

- (i) $L(k) \leq \alpha(k) \leq U(k)$ (correctness);
- (ii) $|[L(k+1), U(k+1)]| \leq |[L(k), U(k)]|$ (non-increasing error);
- (iii) there exists $k_0 > 0$ such that $[L(k), U(k)] \cap \mathcal{U} = \alpha(k)$ for all $k \geq k_0$ (convergence);
- (i′) $z_L(k) \leq z(k) \leq z_U(k)$ (correctness);
- (ii′) $d(z_L(k), z_U(k)) \leq \gamma(|[L(k), U(k)]|)$, with γ a monotonically increasing function of its argument (non-increasing error);
- (iii′) there exists $k_0' > 0$ such that $d(z_{L'}(k), z_{U'}(k)) = 0$ for all $k \geq k_0'$ (convergence), where $L'(k) = \bigwedge([L(k), U(k)] \cap \mathcal{U})$, $U'(k) = \bigvee([L(k), U(k)] \cap \mathcal{U})$, $q_{L'}(k+1) = f_3(q_{L'}(k), L'(k), U'(k), y(k), y(k+1))$, $q_{U'}(k+1) = f_4(q_{U'}(k), L'(k), U'(k), y(k), y(k+1))$, and

$$z_{L'}(k) = \bigwedge g_3\left([q_{L'}(k), q_{U'}(k)] \cap (\mathcal{U} \times \mathcal{Z})\right) \tag{6}$$

$$z_{U'}(k) = \bigvee g_4\left([q_{L'}(k), q_{U'}(k)] \cap (\mathcal{U} \times \mathcal{Z})\right) \tag{7}$$

with $q_{L'}(0) = q_L(0)$ and $q_{U'}(0) = q_U(0)$, for some distance function "d."

Properties (i′)–(iii′) are the same as the properties (i)–(iii) but for the continuous state estimate. The variables L and U represent the lower and upper bounds in $(\chi, \leq)$ of the set of all possible discrete variable values α that are compatible with the output sequence and with the discrete state system dynamics given by Σ_1. The variables z_L and z_U instead represent the lower and upper bounds in $(\mathcal{Z}_E, \leq)$ of the set of all possible continuous variable values that are compatible with the output sequence, with the system dynamics established by Σ, and with the set of possible discrete variable values. The variables q_L and q_U are auxiliary variables that are needed to model the coupling of the continuous and discrete state dynamics. They represent the lower and upper bounds in the mixed discrete-continuous partial order $(\mathcal{L}, \leq)$ of the set of all possible pairs (α, z) compatible with the output sequence, with the system dynamics, and with the set of possible discrete variable values. The distance function "d" has been left unspecified for the moment and can be any function that satisfies the items of Definition 3.1. As we performed in the example in Section 2, in order to construct an estimator that keeps track of lower and upper bounds of the state variables, the state variables of the system have to be viewed as belonging to a partial order. We thus introduce the notion of extension of a system Σ to a partial order.

Definition 4.1. Consider the system $\Sigma = \Sigma_1 \circ_f \Sigma_2$ with $\Sigma_1 = (\mathcal{U}, \mathcal{Y}, \mathcal{U}, f, \text{id}_1)$ and $\Sigma_2 = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, h, g)$. Let $(\chi, \leq)$, $(\mathcal{Z}_E, \leq)$, and $(\mathcal{L}, \leq)$ be partial orders with $\mathcal{U} \subseteq \chi$, $\mathcal{Z} \subseteq \mathcal{Z}_E$, and $\chi \times \mathcal{Z}_E \subseteq \mathcal{L}$. The system extension is defined as $\tilde{\Sigma} = (\mathcal{L}, \mathcal{Y}, \tilde{F}, \tilde{G})$, in which (i) $\tilde{F} : \mathcal{L} \to \mathcal{L}$ with $\tilde{F}|_{\mathcal{U}\times\mathcal{Z}} = (f', h')$ and $\mathcal{L} - (\mathcal{U} \times \mathcal{Z})$ is invariant under $\tilde{F}$; (ii) $\tilde{G} : \mathcal{L} \to \mathcal{Y}$ with $\tilde{G}|_{\mathcal{U}\times\mathcal{Z}} = g'$; (iii) $\tilde{\Sigma}|_{\chi\times\mathcal{Z}_E} = \tilde{\Sigma}_1 \circ_f \tilde{\Sigma}_2$, in which $\tilde{\Sigma}_1 = (\chi, \mathcal{Y}, \chi, \tilde{f}, \text{id}_1)$ and $\tilde{\Sigma}_2 = (\mathcal{Z}_E, \chi, \mathcal{Y}, \tilde{h}, \tilde{g})$, with $\tilde{f}|_{\mathcal{U}\times\mathcal{Y}} = f$, $\tilde{h}|_{\mathcal{Z}\times\mathcal{U}} = h$, and $\tilde{g}|_{\mathcal{Z}\times\mathcal{U}} = g$; (iv) the partial order $(\mathcal{L}, \leq)$ is closed with respect to $\chi \times \mathcal{Z}_E$.

Let $\tilde{\Sigma} = (\mathcal{L}, \mathcal{Y}, \tilde{F}, \tilde{G})$ be the extension of Σ on the lattice $(\mathcal{L}, \leq)$. For all $y_1, y_2 \in \mathcal{Y}$, the sets $T_{y_1,y_2}(\tilde{\Sigma}) = \{w \in \chi \mid \exists\, z \in \mathcal{Z}$ such that $y_2 = \tilde{G}(\tilde{F}(w, z))$ and $y_1 = \tilde{G}(w, z)\}$ are named the *$\tilde{\Sigma}$-transition sets*. The $\tilde{\Sigma}$-transition sets correspond to the set of all possible values of $w \in \chi$ compatible with two consecutive outputs of the extended system $\tilde{\Sigma}$. The *output set* denoted $O_y(k)$ is a transition set corresponding to two consecutive output measurements $(y(k), y(k+1))$ of $\tilde{\Sigma}$ along an execution of $\tilde{\Sigma}$ with output sequence $\{y(k)\}_{k\in\mathbb{N}}$. That is, $O_y(k) := T_{y(k),y(k+1)}(\tilde{\Sigma})$. The next definition introduces the notion of interval compatibility of the tuple $(\tilde{\Sigma}_1, \tilde{\Sigma}, (\chi, \leq))$.

Definition 4.2. The tuple $(\tilde{\Sigma}, \tilde{\Sigma}_1, (\chi, \leq))$ is said to be *interval compatible* if for all $y_1, y_2 \in \mathcal{Y}$, we have that (i) the $\tilde{\Sigma}$-transition sets are intervals, i.e., $T_{y_1,y_2}(\tilde{\Sigma}) = [\bigwedge T_{y_1,y_2}(\tilde{\Sigma}), \bigvee T_{y_1,y_2}(\tilde{\Sigma})]$; (ii) $\tilde{f} : (T_{y_1,y_2}(\tilde{\Sigma}), y_1) \to [\tilde{f}(\bigwedge T_{y_1,y_2}(\tilde{\Sigma}), y_1), \tilde{f}(\bigvee T_{y_1,y_2}(\tilde{\Sigma}), y_1)]$ is an order isomorphism.

This property requires that a $\tilde{\Sigma}$-transition set is a sublattice interval in the lattice $(\chi, \leq)$ and that the extension $\tilde{\Sigma}_1$ is such that $\tilde{f}$ is an order isomorphism on such a set. In order to determine the set of variable values in $\mathcal{L}$ of the extended system that are compatible with an output pair y_1, y_2 and with a set of possible discrete variable values $[w_1, w_2] \subseteq T_{y_1,y_2}(\tilde{\Sigma})$, we introduce the notion of induced output set. Consider the system $\tilde{\Sigma} = (\mathcal{L}, \mathcal{Y}, \tilde{F}, \tilde{G})$ and a transition set $T_{y_1,y_2}(\tilde{\Sigma})$ for some $y_1, y_2 \in \mathcal{Y}$. For all $w_1, w_2 \in T_{y_1,y_2}(\tilde{\Sigma})$ with $w_1 \leq w_2$, the sets $I_{y_1,y_2}^{[w_1,w_2]} = \{q \in \mathcal{L} \mid \pi_1 \circ a_L(q) \geq w_1, \pi_1 \circ a_U(q) \leq w_2, y_2 = \tilde{G}(\tilde{F}(q))$, and $y_1 = \tilde{G}(q)\}$ are named the *induced output sets* of $\tilde{\Sigma}$ induced by an interval $[w_1, w_2] \subseteq T_{y_1,y_2}(\tilde{\Sigma})$. The meaning of an induced output set is the following. The set $I_{y_1,y_2}^{[w_1,w_2]}$ is the set of all possible values of $q \in \mathcal{L}$ that are compatible with two output measurements y_1, y_2 *and* whose upper and lower approximations in $\chi \times \mathcal{Z}_E$ have the discrete component contained in the set $[w_1, w_2]$. One can easily verify that if $[w_1, w_2] \subseteq T_{y_1,y_2}(\tilde{\Sigma})$, then $\{(\alpha, z) \mid g(z, \alpha) = y_1$ and $g(h(z, \alpha), f(\alpha, y_1)) = y_2\}$ with $\alpha \in [w_1, w_2]$ is contained in $I_{y_1,y_2}^{[w_1,w_2]}$. Next, a definition similar to interval compatibility is introduced for the induced output sets and the system extension $\tilde{\Sigma}$.

Definition 4.3. The pair $(\tilde{\Sigma}, (\mathcal{L}, \leq))$ is said to be *induced interval compatible* if for any $[w_1, w_2] \subseteq T_{y_1,y_2}(\tilde{\Sigma})$ for $y_1, y_2 \in \mathcal{Y}$, we have that (i) $\tilde{F} : ([\bigwedge I_{y_1,y_2}^{[w_1,w_2]}, \bigvee I_{y_1,y_2}^{[w_1,w_2]}]) \to [\tilde{F}(\bigwedge I_{y_1,y_2}^{[w_1,w_2]}), \tilde{F}(\bigvee I_{y_1,y_2}^{[w_1,w_2]})]$ is order preserving; (ii) $\tilde{F} : ([\bigwedge I_{y_1,y_2}^{[\alpha,\alpha]}, \bigvee I_{y_1,y_2}^{[\alpha,\alpha]}]) \to [\tilde{F}(\bigwedge I_{y_1,y_2}^{[\alpha,\alpha]}), \tilde{F}(\bigvee I_{y_1,y_2}^{[\alpha,\alpha]})]$ is an order isomorphism; (iii) for all $[w_1, w_2] \subseteq T_{y_1,y_2}(\tilde{\Sigma})$, we have that $d\left(\pi_2 \circ a_L \circ \tilde{F}(\bigwedge I_{y_1,y_2}^{[w_1,w_2]}), \pi_2 \circ a_U \circ \tilde{F}(\bigvee I_{y_1,y_2}^{[w_1,w_2]})\right) \leq \gamma(|[w_1, w_2]|)$, for some distance function "d", and $\gamma : \mathbb{N} \to \mathbb{R}$ a monotonically increasing function of its argument.

The function d is any function that satisfies the items of Definition 3.1. Item (i) of this definition requires that the extended function $\tilde{F}$ has order preserving properties on the induced output sets. Item (ii) requires the stronger property of order isomorphism when the interval to which the discrete state belongs is a singleton. This property is stronger than order preserving because it also requires that $\tilde{F}$ is onto on the indicated codomain. This property is necessary to prove the convergence of the continuous state estimator. Item (iii) establishes that the distance between the lower and upper bounds of the interval sublattice in $(\mathcal{Z}_E, \leq)$ induced by an interval $[w_1, w_2] \in \chi$ is bounded by a monotonic function of the cardinality of $[w_1, w_2]$. When $(y_1, y_2) = (y(k), y(k+1))$ in the above definitions, in which $\{y(k)\}_{k\in\mathbb{N}}$ is an output sequence of Σ, we use the notation $(y(k), y(k+1)) := Y(k)$ so that $I^{[w_1,w_2]}_{y(k),y(k+1)} = I^{[w_1,w_2]}_{Y(k)}$. A solution to Problem 1 is determined on the basis of two intermediate results. The first result establishes that when the tuple $(\tilde{\Sigma}, \tilde{\Sigma}_1, (\chi, \leq))$ is interval compatible, under the assumption of independent discrete state observability it is possible to construct a convergent discrete state estimator $\hat{\Sigma}_1$. The second result establishes that under the induced interval compatibility assumption and given a convergent discrete state estimator, it is possible to construct a convergent continuous state estimator $\hat{\Sigma}_2$ that is driven by the discrete state estimates.

Lemma 4.1. *Let $\{y(k)\}_{k\in\mathbb{N}}$ be the output sequence of an execution of Σ. Consider the system with input $\hat{\Sigma}_1 = (\chi \times \chi, \mathcal{Y} \times \mathcal{Y}, \chi \times \chi, (f_1, f_2), id)$ with $f_1 : \chi \times \mathcal{Y} \times \mathcal{Y} \to \chi$, $f_2 : \chi \times \mathcal{Y} \times \mathcal{Y} \to \chi$ given by $f_1(L(k), y(k), y(k+1)) = \tilde{f}(\bigwedge O_y(k) \curlyvee L(k), y(k))$ and $f_2(U(k), y(k), y(k+1)) = \tilde{f}(\bigvee O_y(k) \curlywedge U(k), y(k))$, with $L(0) = \bigwedge \chi$ and $U(0) = \bigvee \chi$. If system Σ is independently discrete state observable and the tuple $(\tilde{\Sigma}_1, \tilde{\Sigma}, (\chi, \leq))$ is interval compatible, then $L(k)$ and $U(k)$ have properties* (i)– (iii) *of Problem 1.*

The proof of (i) relies on the order preserving property of $\tilde{f}$. The proof of (ii) exploits the order isomorphism property of $\tilde{f}$ on the output set. The proof of (iii) relies on the independent discrete state observability assumption. For the details, the reader is referred to [10].

Lemma 4.2. *Let $\{y(k)\}_{k\in\mathbb{N}}$ be an output sequence of Σ. Let $\hat{\Sigma}_1$ be as in Lemma 4.1 and let the hypotheses of Lemma 4.1 be satisfied. Consider the system with input $\hat{\Sigma}_2 = (\mathcal{L} \times \mathcal{L}, \chi \times \chi \times \mathcal{Y} \times \mathcal{Y}, \chi \times \chi \times \mathcal{Z}_E \times \mathcal{Z}_E, (f_3, f_4), (g_1, g_2, g_3, g_4))$ with $f_3 : \mathcal{L} \times \chi \times \chi \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{L}$, $f_4 : \mathcal{L} \times \chi \times \chi \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{L}$, $g_3 : \mathcal{L} \to \mathcal{Z}_E$, and $g_4 : \mathcal{L} \to \mathcal{Z}_E$ given by*

$$f_3(q_L(k), L(k), U(k), y(k), y(k+1)) = \tilde{F}\left(q_L(k) \curlyvee \bigwedge I^{[L^*(k),U^*(k)]}_{Y(k)}\right)$$

$$f_4(q_U(k), L(k), U(k), y(k), y(k+1)) = \tilde{F}\left(q_U(k) \curlywedge \bigvee I^{[L^*(k),U^*(k)]}_{Y(k)}\right)$$

$$g_3(q_L(k)) = \pi_2 \circ a_L(q_L(k)), \qquad g_4(q_U(k)) = \pi_2 \circ a_U(q_U(k)) \tag{8}$$

in which $L(k), U(k), y(k), y(k+1)$ is the output of $\hat{\Sigma}_1$, $L^*(k) = \bigwedge O_y(k) \curlyvee L(k)$, and $U^*(k) = \bigvee O_y(k) \curlywedge U(k)$. If $(\tilde{\Sigma}, (\mathcal{L}, \leq))$ is induced interval compatible and Σ is observable, then $z_L(k) = g_3(q_L(k))$ and $z_U(k) = g_4(q_U(k))$ have properties (i′)–(iii′) *of Problem 1.*

Proof. The proof of (i′) exploits the order preserving properties of $\tilde{F}, a_L, a_U$, and of π_2. The proof of (ii′) exploits the property of induced order compatibility and the definition of distance on a partial order. The proof of (iii′) uses directly the observability of system Σ.

Proof of (i′). We use induction argument on k. Initially, $q_L(0) = \bigwedge \mathcal{L}$ and $q_U(0) = \bigvee \mathcal{L}$. Therefore we have that $q_L(0) \leq (\alpha(0), z(0)) \leq q_U(0)$. Next, we show that $q_L(k) \leq (\alpha(k), z(k)) \leq q_U(k)$ implies $q_L(k+1) \leq (\alpha(k+1), z(k+1)) \leq q_U(k+1)$. Since $\alpha(k) \in [L^*(k), U^*(k)] \subseteq T_{y(k),y(k+1)}(\tilde{\Sigma})$ and $(\alpha(k), z(k)) \in \{(\alpha, z) \mid g(z, \alpha) = y(k)$ and $g(h(z, \alpha), f(\alpha, y(k))) = y(k+1)\}$ we have that $(\alpha(k), z(k)) \in I^{[L^*(k),U^*(k)]}_{Y(k)}$. Removing the dependency of L^*, U^* and Y on k, we obtain that $q_L(k) \curlyvee \bigwedge I^{[L^*,U^*]}_Y \leq (\alpha(k), z(k)) \leq q_U(k) \curlywedge \bigvee I^{[L^*,U^*]}_Y$. Since $\tilde{F}$ is order preserving on $I^{[L^*,U^*]}_Y$, we also have that $\tilde{F}\left(q_L(k) \curlyvee \bigwedge I^{[L^*,U^*]}_Y\right) \leq (\alpha(k+1), z(k+1)) \leq \tilde{F}\left(q_U(k) \curlywedge \bigvee I^{[L^*,U^*]}_Y\right)$. We are left to show that $q_L(k) \leq (\alpha(k), z(k))$ ($(\alpha(k), z(k)) \geq q_L(k)$) implies that $\pi_2 \circ a_L(q_L(k)) \leq z(k)$ ($\pi_2 \circ a_U(q_L(k)) \geq z(k)$). This is true as $\pi_2 \circ a_L$ ($\pi_2 \circ a_U$) is an order preserving map and $\pi_2 \circ a_L(\alpha(k), z(k)) = z(k)$ ($\pi_2 \circ a_U(\alpha(k), z(k)) = z(k)$).

Proof of (ii′). Since $\tilde{F}$ is order preserving on the induced transition sets, we have that $\tilde{F}(\bigwedge I^{[L^*,U^*]}_Y) \leq \tilde{F}(q_L(k) \curlyvee \bigwedge I^{[L^*,U^*]}_Y)$ ($\tilde{F}(\bigvee I^{[L^*,U^*]}_Y) \geq \tilde{F}(q_U(k) \curlywedge \bigvee I^{[L*,U^*]}_Y)$). Since $\pi_2 \circ a_L$ and $\pi_2 \circ a_U$ are also order preserving, by using property (iii) of the distance function, we have that $d(\pi_2 \circ a_L \circ \tilde{F}(q_L(k) \curlyvee \bigwedge I^{[L^*,U^*]}_Y), \pi_2 \circ a_U \circ \tilde{F}(q_U(k) \curlywedge \bigvee I^{[L^*,U^*]}_Y)) \leq d(\pi_2 \circ a_L \circ \tilde{F}(\bigwedge I^{[L^*,U^*]}_Y), \pi_2 \circ a_U \circ \tilde{F}(\bigvee I^{[L^*,U^*]}_Y))$. By property (iii) of Definition 4.3, we have that $d(\pi_2 \circ a_L \circ \tilde{F}(\bigwedge I^{[L^*,U^*]}_Y), \pi_2 \circ a_U \circ \tilde{F}(\bigvee I^{[L^*,U^*]}_Y)) \leq \gamma([L^*, U^*])$. Since $\tilde{f}([L^*, U^*], y) \subseteq [\tilde{f}(L^*, y), \tilde{f}(U^*, y)]$, $\tilde{f}(L^*(k), y(k)) = L(k+1)$, and $\tilde{f}(U^*(k), y(k)) = U(k+1)$, we have by the order isomorphism property of $\tilde{f}$ that $|\tilde{f}([L^*(k), U^*(k)], y(k))| = |[L^*(k), U^*(k)]| \leq |[L(k+1), U(k+1)]|$. Since γ is a monotonically increasing function of its argument, we have that $\gamma(|[L^*(k), U^*(k)]|) \leq \gamma(|[L(k+1), U(k+1)]|)$.

Proof of (iii′). For $k > k_0$, $L'(k) = \alpha(k) = U'(k)$ because $[L(k), U(k)] \cap \mathcal{U} = \alpha(k)$. As a consequence, $q_{L'}(k+1) = \tilde{F}(q_{L'}(k) \curlyvee \bigwedge I^{[\alpha(k),\alpha(k)]}_{Y(k)})$ and $q_{U'}(k+1) = \tilde{F}(q_{U'}(k) \curlywedge \bigvee I^{[\alpha(k),\alpha(k)]}_{Y(k)})$. By property (ii) of Definition 4.3, it follows that for all $k > k_0$ we have that for all $q' \in [q_{L'}(k+1), q_{U'}(k+1)]$ there is $q \in [q_{L'}(k), q_{U'}(k)]$ such that $q' = \tilde{F}(q)$. Also, $\mathcal{L} - (\mathcal{U} \times \mathcal{Z})$ is invariant under $\tilde{F}$ and $\tilde{F}|_{\mathcal{U}\times\mathcal{Z}} = (f', h')$, Therefore, it is also true that for all $(\alpha', z') \in [q_{L'}(k+1), q_{U'}(k+1)] \cap (\mathcal{U} \times \mathcal{Z})$ there is $(\alpha, z) \in [q_L(k), q_U(k)] \cap (\mathcal{U} \times \mathcal{Z})$ such that $(\alpha', z') = (f(\alpha, y(k)), h(z, \alpha))$. In addition, we have that such (α, z) is in the induced transition set, that is, $(\alpha, z) \in I^{[\alpha(k),\alpha(k)]}_{Y(k)}$. This in turn implies that $g(z, \alpha) = y(k)$. This is true for all $k \geq k_0$. But, if for all $k \geq k_0$ the set $[q'_L(k), q'_U(k)] \cap (\mathcal{U} \times \mathcal{Z})$ contains more than one element, it means that there are at least two executions of Σ, $\sigma_1 \neq \sigma_2$, such that $g(\sigma_1) = g(\sigma_2)$. This contradicts observability of Σ. Thus, it must be that there is $k'_0 > k_0$ such that for $k \geq k'_0$ we have that $[q'_L(k), q'_U(k)] \cap (\mathcal{U} \times \mathcal{Z}) = (\alpha(k), z(k))$. As a consequence, by virtue of Eq. (8) we also have that $z_{L'}(k) = z_{U'}(k)$ for all $k \geq k'_0$. □

The proof of the convergence of the continuous state estimate relies on the fact that the function $\tilde{F}$ is an order isomorphism. If the extended function $\tilde{F}$ is not an order isomorphism, the convergence of the continuous state cannot be guaranteed. While the proof of the convergence of the continuous state estimates relies on the convergence of the discrete state estimates, it is not necessary that the continuous state estimate awaits the convergence of the discrete state estimate before it can converge. The discrete and continuous state estimates can converge at the same time. This is due to the fact that the estimation strategy (both for the continuous and discrete states) relies on a prediction-correction approach. Due to this approach, the error on the continuous variable estimates can be rendered smaller at each step. The following theorem is a consequence of Lemmas 4.1 and 4.2.

Theorem 4.1. *The cascade interconnection $\hat{\Sigma} = \hat{\Sigma}_1 \circ_c \hat{\Sigma}_2$, where $\hat{\Sigma}_1$ is as in Lemma 4.1 and $\hat{\Sigma}_2$ is as in Lemma 4.2, solves Problem 1.*

We next study a class of systems in which there is a partial order on $\mathcal{Z}$, the cone partial order, that is preserved by the system dynamics. In this case, we can choose $\mathcal{Z}_E = \mathcal{Z}$, with $(\mathcal{Z}, \leq)$ a partial order, and $(\mathcal{L}, \leq) = (\chi \times \mathcal{Z}, \leq)$. An *ordered Banach space* [11] is a real Banach space $\mathcal{Z}$ with a non-empty closed subset K known as the positive cone with the following properties: (i) $\alpha K \subseteq K$ for all $\alpha \in \mathbb{R}_+$; (ii) $K + K \subseteq K$; (iii) $K \cap (-K) = \{\emptyset\}$, i.e., the cone is pointed. A partial ordering is then defined by $x \geq y$ for all $x, y \in \mathcal{Z}$ if and only if $x - y \in K$, with $x > y$ if and only if $x \geq y$ and $x \neq y$. The resulting partial order is denoted $(\mathcal{Z}, \leq)$. If condition (iii) is not satisfied, we simply refer to $(\mathcal{Z}, \leq)$ as an ordered space. Let again $\Sigma = \Sigma_1 \circ_f \Sigma_2$, in which $\Sigma_1 = (\mathcal{U}, \mathcal{Y}, \mathcal{U}, f, \text{id})$ and $\Sigma_2 = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, h, g)$ with $(\mathcal{Z}, \leq)$ an ordered space. Let $(\chi, \leq)$ be a lattice and consider the extension $\tilde{\Sigma} = \tilde{\Sigma}_1 \circ_f \tilde{\Sigma}_2$, in which $\tilde{\Sigma}_1 = (\chi, \mathcal{Y}, \chi, \tilde{f}, \text{id})$ and $\tilde{\Sigma}_2 = (\mathcal{Z}, \chi, \mathcal{Y}, \tilde{h}, \tilde{g})$, with $\tilde{f} : \chi \times \mathcal{Y} \to \chi$ and $\tilde{f}|_{\mathcal{U}\times\mathcal{Y}} = f$; $\tilde{h} : \mathcal{Z} \times \chi \to \mathcal{Z}$ with $\tilde{h}|_{\mathcal{Z}\times\mathcal{U}} = h$; $\tilde{g} : \mathcal{Z} \times \chi \to \mathcal{Y}$ and $\tilde{g}|_{\mathcal{Z}\times\mathcal{U}} = g$. Then, we say that Σ is a *monotone deterministic transition system* if there is a lattice $(\chi, \leq)$ with $\mathcal{U} \subseteq \chi$ and a system extension $\tilde{\Sigma} = \tilde{\Sigma}_1 \circ_f \tilde{\Sigma}_2$ on $(\chi \times \mathcal{Z}, \leq)$ such that $\tilde{h}$ is order preserving. Such an extension $\tilde{\Sigma}$ is called a *monotone extension* of Σ on $\chi \times \mathcal{Z}$. For a monotone deterministic transition system, the ordered space $(\mathcal{Z}, \leq)$ can be used in the estimator design to reduce the computational burden, as the elements of $\mathcal{Z}$ are points, and their order relation can be efficiently computed by using the definition of $(\mathcal{Z}, \leq)$. In the following, we assume that $(\mathcal{Z}, \leq)$ in an ordered Banach space unless otherwise stated. For a monotone transition system, we can re-define the induced output sets to contain only the continuous component of the state, that is, for all $y_1, y_2 \in \mathcal{Y}$ and $w_1 \leq w_2 \in T_{y_1,y_2}(\tilde{\Sigma})$ we define $I_{y_1,y_2}^{[w_1,w_2]} := \{z \in \mathcal{Z} \mid y_1 = \tilde{g}(z, w), y_2 = \tilde{g}(\tilde{h}(z, w), \tilde{f}(w, y_1)), w \in [w_1, w_2]\}$. In addition, the induced order compatibility definition is defined only on the basis of the properties of $\tilde{h}$. This implies that items (i)–(iii) of Definition 4.3 take the form: (i) $\tilde{h} : I_{y_1,y_2}^{[w_1,w_2]} \times [w_1, w_2] \to [\tilde{h}(\bigwedge I_{y_1,y_2}^{[w_1,w_2]}, w_1), \tilde{h}(\bigvee I_{y_1,y_2}^{[w_1,w_2]}, w_2)]$ is order preserving; (ii) $\tilde{h} : I_{y_1,y_2}^{[\alpha,\alpha]} \times \alpha \to [\tilde{h}(\bigwedge I_{y_1,y_2}^{[\alpha,\alpha]}, \alpha), \tilde{h}(\bigvee I_{y_1,y_2}^{[\alpha,\alpha]}, \alpha)]$ is an order isomorphism; (iii) $d(\tilde{h}(\bigwedge I_{y_1,y_2}^{[w_1,w_2]}, w_1), \tilde{h}(\bigvee I_{y_1,y_2}^{[w_1,w_2]}, w_2)) \leq \gamma(|[w_1, w_2]|)$. For a monotone deterministic transition system, induced interval compatibility can be easily verified and the values of $\bigvee I_{y_1,y_2}^{[w_1,w_2]}$ and $\bigwedge I_{y_1,y_2}^{[w_1,w_2]}$ can be efficiently computed. A map M can be found that for each pair of consecutive output observations attaches to the discrete state a value of the continuous state. If this map is order preserving, then the values of the ends of the interval induced by $[w_1, w_2]$ can be simply computed by computing the map M on w_1 and on w_2. In general, let $\{y(k)\}_{k\in\mathbb{N}}$ be an output sequence of Σ. Define $\tilde{h}^k(z, w) := \tilde{h}(\tilde{h}^{k-1}(z, w), \tilde{f}^{k-1}(w, y(k-2)))$, and $\tilde{f}^k(w, y(k-1)) := \tilde{f}(\tilde{f}^{k-1}(w, y(k-2), y(k-1)))$, with $\tilde{f}^0(w, y) := w$ and $\tilde{h}^0(z, w) := z$.

Definition 4.4. Let Σ be a monotone transition system and $\tilde{\Sigma} = \tilde{\Sigma}_1 \circ_f \tilde{\Sigma}_2$ its monotone extension on the partial order $(\chi \times \mathcal{Z}, \leq)$, with $(\mathcal{Z}, \leq)$ an ordered space. We say that $\tilde{\Sigma}$ is *continuous state observable in* $\bar{k}$ *steps* if there is $\bar{k} > 0$ such that for all $k \geq 0$, for all output sequences of $\Sigma\{y(k)\}_{k\geq 0}$, and for all $w \in \chi$ we have that $\{z \mid \tilde{g}(z, w) = y(k), \ldots, \tilde{g}(\tilde{h}^{\bar{k}-1}(z, w)), \tilde{f}^{\bar{k}-1}(w, y(k+\bar{k}-2)) = y(k+\bar{k}-1)\}$ is a singleton in $\mathcal{Z}$. The map $M_{Y(k)} : \chi \to \mathcal{Z}$ that for each finite output sequence $Y(k) = \{y(i)\}_{i\in[k,k+\bar{k}-1]}$ attaches to a $w \in \chi$ the $z \in \mathcal{Z}$ according to $M_{Y(k)}(w) := \{z \mid \tilde{g}(z, w) = y(k), \ldots, \tilde{g}(\tilde{h}^{\bar{k}-1}(z, w)), \tilde{f}^{\bar{k}-1}(w, y(k+\bar{k}-2)) = y(k+\bar{k}-1)\}$ is the *observability map*.

Thus, if the system $\tilde{\Sigma}$ is continuous state observable, the continuous state z can be expressed as a function of the output sequence and of a starting discrete state $w \in \chi$.

Proposition 1. *If the monotone extension of* Σ*,* $\tilde{\Sigma}$*, is continuous state observable in two steps, then* $(\tilde{\Sigma}, (\chi \times \mathcal{Z}, \leq))$*, with* $(\mathcal{Z}, \leq)$ *an ordered Banach space, is induced interval compatible. Furthermore, let* $\{y(k)\}_{k\in\mathbb{N}}$ *be an output sequence of* Σ*, if the observability map* $M_{Y(k)} : \chi \to \mathcal{Z}$ *is also order preserving, then for all* $w_1 \leq w_2$ *with* $w_1, w_2 \in O_y(k)$ *we have that* $\bigwedge I_{Y(k)}^{[w_1,w_2]} = M_{Y(k)}(w_1)$ *and* $\bigvee I_{Y(k)}^{[w_1,w_2]} = M_{Y(k)}(w_2)$*.*

Proof. Item (i) of Definition 4.3 is satisfied as $\tilde{h}$ is order preserving due to the fact that $\tilde{\Sigma}$ is a monotone extension of Σ on $(\chi \times \mathcal{Z}, \leq)$. Item (ii) of Definition 4.3 is clearly verified as $\bigwedge I_{y_1,y_2}^{[\alpha,\alpha]} = \bigvee I_{y_1,y_2}^{[\alpha,\alpha]}$ by the assumption of continuous state observability in two steps. Let $\bar{d} := \max_{w_i \ll w_j} \|\tilde{h}(M_{Y(k)}(w_i), w_i) - \tilde{h}(M_{Y(k)}(w_j), w_j)\|$ for $w_i, w_j \in [w_1, w_2] \subseteq \chi$, then (iii) is verified with $\gamma(|[w_1, w_2]|) = \bar{d}|[w_1, w_2]|$ by using the triangular inequality. By the hypothesis of observability in two steps, it follows that $\bigwedge I_{Y(k)}^{[w,w]} = z^* = \bigvee I_{Y(k)}^{[w,w]} = M_{Y(k)}(w)$. By the order preserving property of $M_{Y(k)}$, it follows that $M_{Y(k)}(w_1) \leq M_{Y(k)}(w_2)$ when $w_1 \leq w_2$. □

There is an extensive literature in the context of monotone systems that studies conditions for the monotonicity of maps. The reader is referred to [11] for details. A monotone system extension that is continuous state observable in two steps automatically satisfies the induced interval compatibility properties. Also, the lower and upper bounds of the induced output sets that are used in the estimator update laws can be readily computed if the observability map is order preserving. As a consequence, for a monotone extension $\tilde{\Sigma}$ for which the observability map is order preserving, the update laws f_3 and f_4 of Theorem 4.1 transform to

$$f_3(z_L(k), L(k), U(k), y(k), y(k+1)) = \tilde{h}\left(z_L(k) \curlyvee M_{Y(k)}(L^*(k)), L^*(k)\right)$$
$$f_4(z_U(k), L(k), U(k), y(k), y(k+1)) = \tilde{h}\left(z_U(k) \curlywedge M_{Y(k)}(U^*(k)), U^*(k)\right). \quad (9)$$

5. Estimator Existence

Theorem 5.1. *Assume that the system* $\Sigma = \Sigma_1 \circ_f \Sigma_2$ *is observable and independently discrete state observable. Then there exist lattices* $(\chi, \leq)$*,* $(\mathcal{Z}_E, \leq)$*,* $(\mathcal{L}, \leq)$ *with* $\mathcal{U} \subseteq \chi$*,* $\mathcal{Z} \subseteq \mathcal{Z}_E$*, and* $\chi \times \mathcal{Z}_E \subseteq \mathcal{L}$*, and system extensions* $\tilde{\Sigma}_1$ *and* $\tilde{\Sigma}$ *such that the tuple* $(\tilde{\Sigma}_1, \tilde{\Sigma}, (\chi, \leq))$ *is interval compatible and* $(\tilde{\Sigma}, (\mathcal{L}, \leq))$ *is induced interval compatible.*

Proof. In the first part of the proof, we construct the lattice $(\chi, \leq)$, we define $\tilde{\Sigma}_1$ and $\tilde{\Sigma}$ on $\chi \times \mathcal{Z}$. Then we show that properties (i)–(iii) of Definition 4.2 are verified. In the second part of the proof, we define the lattices $(\mathcal{L}, \leq)$, $(\mathcal{Z}_E, \leq)$, we define $\tilde{\Sigma}$ on $\mathcal{L}$, and we define an appropriate distance function d on $\mathcal{Z}_E$. Thus, we show that properties (i)–(iii) of Definition 4.3 are satisfied.

Part 1. We define the lattice $(\chi, \leq)$ as $(\chi, \leq) = (\mathcal{P}(\mathcal{U}), \subseteq)$, that is, the set of all subsets of $\mathcal{U}$ with order established according to set inclusion. The bottom element is the empty set, denoted $\perp$, and the top element is $\mathcal{U}$ itself. Any element in χ, denoted w, is of the form $w = \alpha_1 \curlyvee \cdots \curlyvee \alpha_n$ for some $\alpha_i \in \mathcal{U}$. The system $\tilde{\Sigma}_1 = (\chi, \mathcal{Y}, \chi, \tilde{f}, \text{id})$ is determined once $\tilde{f} : \chi \times \mathcal{Y} \to \chi$ is established. The function $\tilde{f} : \chi \times \mathcal{Y} \to \chi$ is defined as $\tilde{f}(w, y) = f(\alpha_1, y) \curlyvee \cdots \curlyvee f(\alpha_n, y)$ for any $w = \alpha_1 \curlyvee \cdots \curlyvee \alpha_n$, and $\tilde{f}(\perp, y) = \perp$, for any $y \in \mathcal{Y}$. In order to identify the structure of the sets $T_{y_1,y_2}(\tilde{\Sigma})$, we define $\tilde{\Sigma} = (\mathcal{L}, \mathcal{Y}, \tilde{F}, \tilde{G})$ by initially defining $\tilde{F}$ and $\tilde{G}$ on $\chi \times \mathcal{Z}$. Recall that $\tilde{G}|_{\chi\times\mathcal{Z}_E} = \tilde{g}'$ and that $\tilde{g}'(w, \bar{z}) = \tilde{g}(\bar{z}, w)$ for all $w \in \chi$ and $\bar{z} \in \mathcal{Z}_E$. For all $(w, z) \in \chi \times \mathcal{Z}$, for $w = \alpha_1 \curlyvee \cdots \curlyvee \alpha_n$, we define $\tilde{F}(w, z) := F(\alpha_1, z) \curlyvee \cdots \curlyvee F(\alpha_n, z)$, $F(\alpha, z) := (f(\alpha, y), h(z, \alpha))$, with $y = g(z, \alpha)$ for any $\alpha \in \mathcal{U}$. Also, we define the function $\tilde{g}(z, w)$ for all $(w, z) \in \chi \times \mathcal{Z}$ with $w = \alpha_1 \curlyvee \cdots \curlyvee \alpha_n$ as follows. We set $\tilde{g}(z, w) = y$ if and only if $g(z, \alpha_i) = y$ for all i. As a consequence, one can check that if $T_{y_1,y_2}(\Sigma) = \{\alpha_1, \ldots, \alpha_n\}$ for some $\alpha_i \in \mathcal{U}$, then it follows that $T_{y_1,y_2}(\tilde{\Sigma}) = [\perp, \alpha_1 \curlyvee \cdots \curlyvee \alpha_n]$. This directly follows from the definition of $\tilde{F}$ and $\tilde{g}$. From this fact, it follows that (i) of Definition 4.2 holds. Property (ii) follows from the fact that $\tilde{f}$ is an order embedding by definition, from the fact that $\tilde{f} : [\perp, w] \to [\perp, \tilde{f}(w, y)]$ is onto by definition, and from the fact that $\tilde{f} : T_{y_1,y_2}(\tilde{\Sigma}) \to \tilde{f}(T_{y_1,y_2}(\tilde{\Sigma}), y_1)$ must be one–one by the independent discrete state observability assumption.

Part 2. In the second part of the proof, lattices $(\mathcal{Z}_E, \leq)$, and $(\mathcal{L}, \leq)$ with extensions $\tilde{F} : \mathcal{L} \to \mathcal{L}$ and $\tilde{G} : \mathcal{L} \to \mathcal{Y}$ are constructed. Define $\{z \mid y = g(z, \alpha), \alpha \in \mathcal{U}\} := m(\alpha, y)$. The set $\mathcal{Z}_E$ is defined in the following way: (i) $\mathcal{Z} \subseteq \mathcal{Z}_E$; (ii) $m(\alpha, y) \in \mathcal{Z}_E$ for any $y \in \mathcal{Y}$ and any $\alpha \in \mathcal{U}$; (iii) $\mathcal{Z}_E$ is invariant under h, i.e., if $\bar{z} \in \mathcal{Z}_E$, then $h(\bar{z}, \alpha) \in \mathcal{Z}_E$ for all $\bar{z} \in \mathcal{Z}_E$ and all $\alpha \in \mathcal{U}$; (iv) $\mathcal{Z}_E$ is closed under finite unions and finite intersections. By construction, $(\mathcal{Z}_E, \leq)$ is a lattice where the order is established by set inclusion. Each element in $\mathcal{Z}_E$ is a union of submanifolds or of points in $\mathcal{Z}$. We denote an element in $\mathcal{Z}_E$ by $\bar{z}$. Define $(\mathcal{L}, \leq) := (\mathcal{P}(\chi \times \mathcal{Z}_E), \subseteq)$, that is, the set of all subsets of $\chi \times \mathcal{Z}_E$ with order established by set inclusion. By construction, $\chi \times \mathcal{Z}_E \subseteq \mathcal{L}$. An element in $\mathcal{L}$ is denoted by $q \in \mathcal{L}$ and it has the form $q = (w_1, \bar{z}_1) \curlyvee \cdots \curlyvee (w_k, \bar{z}_k)$, in which $\bar{z}_i \in \mathcal{Z}_E$ and $w_i \in \chi$. For all $q = (w_1, \bar{z}_1) \curlyvee \cdots \curlyvee (w_k, \bar{z}_k) \in \mathcal{L}$, its lower and its upper approximations are defined as $a_L(q) := (w_1 \curlywedge \cdots \curlywedge w_k, \bar{z}_1 \curlywedge \cdots \curlywedge \bar{z}_k)$ and $a_U(q) := (w_1 \curlyvee \cdots \curlyvee w_k, \bar{z}_1 \curlyvee \cdots \curlyvee \bar{z}_k)$. For all $\bar{z} \in \mathcal{Z}_E$ and for all $w \in \chi$, we define $(\perp, \bar{z}) = (w, \perp) = \perp$. Define the functions $\tilde{G} : \mathcal{L} \to \mathcal{Y}$ and $\tilde{F} : \mathcal{L} \to \mathcal{L}$ in the following way. For all $q = (w_1, \bar{z}_1) \curlyvee \cdots \curlyvee (w_k, \bar{z}_k) \in \mathcal{L}$, with $w_i = \alpha_{i,1} \curlyvee \cdots \curlyvee \alpha_{i,p_i}$, define $\tilde{G}(q) := y$ if and only if $\tilde{g}(\bar{z}_i, w_i) = y$, $\tilde{g}(\bar{z}_i, w_i) := y$ if and only if $g(\bar{z}_i, \alpha_{i,j}) = y$ for all $j \le p_i g(\bar{z}_i, \alpha) = y$ if and only if $\bar{z}_i \subseteq m(\alpha, y)$. Also, define $\tilde{F}(q) := \tilde{F}(w_1, \bar{z}_1) \curlyvee \cdots \curlyvee \tilde{F}(w_k, \bar{z}_n)$, $\tilde{F}(w_i, \bar{z}_i) := \tilde{F}(\alpha_{i,1}, \bar{z}_i) \curlyvee \cdots \curlyvee \tilde{F}(\alpha_{i,p_i}, \bar{z}_i)$, and $\tilde{F}(\alpha, \bar{z}) := (f(\alpha, y), h(\bar{z}, \alpha))$ if $g(\bar{z}, \alpha) = y, y \in \mathcal{Y}, \alpha \in \mathcal{U}, \bar{z} \in \mathcal{Z}_E$, while $\tilde{F}(\alpha, \bar{z}) := \perp$ if $g(\bar{z}, \alpha) \notin \mathcal{Y}, \alpha \in \mathcal{U}, \bar{z} \in \mathcal{Z}_E$. We define $\tilde{F}(\perp) = \perp$. Note that $\bar{z} \subseteq \mathcal{Z}$ and therefore h does not need to be extended. We next determine the structure of the induced transition sets. The intervals $[L^*, U^*]$ in the estimator in Theorem 4.1 always have the lower bound equal to $\perp$ as the output sets $O_y(k)$ have the lower bound equal to $\perp$ and $\tilde{f}(\perp, y) = \perp$. As a consequence, we are interested in the structure of $I^{[\perp,w]}_{y_1,y_2}$ for $[\perp, w] \subseteq T_{y_1,y_2}(\tilde{\Sigma})$. Let $w = \alpha_1 \curlyvee \cdots \curlyvee \alpha_n$. One can then check that $I^{[\perp,w]}_{y_1,y_2} = [\perp, q]$, where $q = (\alpha_1, \bar{z}_1) \curlyvee \cdots \curlyvee (\alpha_n, \bar{z}_n)$, in which $\bar{z}_i \subseteq m(\alpha_i, y_1)$. For all $q \in I^{[\perp,w]}_{y_1,y_2}$ with $q \ne \perp$, the definition of $\tilde{F}$ guarantees that $\tilde{F}$ is order preserving on the induced output set. Thus, (i) of Definition 4.3 is satisfied. To check that also (ii) of the same definition is satisfied, note that $I^{[\alpha,\alpha]}_{y_1,y_2} = (\alpha, \bar{z})$ for $\bar{z} \subseteq m(\alpha, y_1)$. The assumption of Σ being observable and the fact that $\tilde{F}(\alpha, m(\alpha, y_1)) = (f(\alpha, y_1), h(m(\alpha, y_1), \alpha))$ guarantee that $\tilde{F} : (\alpha, m(\alpha, y_1)) \to \tilde{F}(\alpha, m(\alpha, y_1))$ is one–one. Since $\tilde{F}$ is also an order embedding by definition, property (ii) of Definition 4.3 is also verified. To show (iii) of Definition 4.3, we define a distance function on $\mathcal{Z}_E$. For all $\bar{z}_1, \bar{z}_2 \in \mathcal{Z}_E$, we choose the discrete metric, that is, $d(\bar{z}_1, \bar{z}_2) = 1$ if $\bar{z}_1 \ne \bar{z}_2$, $d(\bar{z}_1, \bar{z}_2) = 0$ otherwise. This distance satisfies the definition of a distance on a partial order. In fact, if $x \le y \le z$ we must have that $d(x, y) \le d(x, z)$ otherwise we would have $d(x, z) = 0$ and $d(x, y) = 1$, which is a contradiction. For all $[\perp, w] \subseteq \chi$ with $w \ne \perp$, $|[\perp, w]| \ge 1$ and therefore (iii) of Definition 4.3 is verified with $\gamma = \text{id}$. □

The construction of the partial orders in this theorem is non-trivial due to the mixed discrete–continuous nature of the partial order $(\mathcal{L}, \le)$. In particular, its elements are sets of pairs of elements in χ and in $\mathcal{Z}_E$ and thus they do not necessarily lie in the Cartesian product $\chi \times \mathcal{Z}_E$. Therefore, the order between any two elements in $\mathcal{L}$ cannot simply be derived by the ordering of the Cartesian product $(\chi \times \mathcal{Z}_E, \le)$. This is illustrated in Example 1 in Section 6.

6. Examples

Example 1 (*Linear Discrete-Time Hybrid Automaton*). Let $\mathcal{U} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, and $\alpha(k+1) = f(\alpha(k))$ where f is defined as $f(\alpha_1) = \alpha_4, f(\alpha_2) = \alpha_5, f(\alpha_3) = \alpha_1, f(\alpha_4) = \alpha_5, f(\alpha_5) = \alpha_3$. Assume $\mathcal{Z} = \mathbb{R}^n$, the function h is given by $z(k+1) = A(\alpha(k))z(k) + B(\alpha(k))$, where $A(\alpha_i) = A_i \in \mathbb{R}^n \times \mathbb{R}^n$ and $B(\alpha_i) = B_i \in \mathbb{R}^n$. The output function g is such that $g(z, \alpha) = (g_\alpha(\alpha), g_z(\alpha, z))$, where $g_\alpha : \mathcal{U} \to \{y_1, y_2\}$ and $g_z(\alpha, z) = C(\alpha)z$, with $C(\alpha_i) = C_i \in \mathbb{R}^m \times \mathbb{R}^n$. Thus, $\mathcal{Y} = \{y_1, y_2\} \times \mathbb{R}^m$. We denote the sets $Y_1 = \{\alpha \in \mathcal{U} \mid g_\alpha(\alpha) = y_1\} := \{\alpha_1, \alpha_2, \alpha_3\}$ and $Y_2 = \{\alpha \in \mathcal{U} \mid g_\alpha(\alpha) = y_2\} := \{\alpha_4, \alpha_5\}$. An instance of such an example is considered with $n = 3$ and $m = 1$, in which $A_1 = ((1, 1, 1)', (0, 1, 1)', (0, 0, 1)')'$, $A_2 = ((1/2, 1/2, 1/2)', (1, 2, 2)', (0, 0, 1)')'$, $A_3 = ((2, 1, 1)', (0, 1, 1)', (2, 0, 0)')'$, $A_4 = ((1, 1, 1)', (1, 1, 0)', (0, 0, 1)')'$, $A_5 = ((1, 0, 0)', (1, 1, 1)', (1, 1, 0)')'$, $C_1 = (1, 0, 0)$, $C_2 = (1, 1, 2)$, $C_3 = (0, 0, 0)$, $C_4 = (1, 0, 0)$, and $C_5 = (0, 1, 1)$. In this example, $\Sigma = \Sigma_1 \circ_c \Sigma_2$ with $\Sigma_1 = (\mathcal{U}, \mathcal{U}, f, \text{id})$ and $\Sigma_2 = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, h, g)$. The system $\tilde{\Sigma}_1 = (\chi, \chi, \tilde{f}, \text{id})$ is defined as follows.

The lattice $(\chi, \le)$ whose construction is sketched in Theorem 5.1 is shown in Fig. 2. It contains the set of all subsets of $\mathcal{U}$ on which the estimator can evolve. Such a diagram also shows the way $\tilde{f} : \chi \to \chi$ is defined. The system $\tilde{\Sigma} = (\mathcal{L}, \mathcal{Y}, \tilde{F}, \tilde{G})$ is defined as follows. The lattice $(\mathcal{Z}_E, \le)$ is constructed according to the proof of Theorem 5.1, in which the submanifolds are affine linear subspace. The partial order $(\mathcal{L}, \le)$, the function $\tilde{F}$, and the function $\tilde{G}$ are also constructed as in the proof of Theorem 5.1. An element of $\mathcal{L}$ is a set of pairs (α_i, m_i), in which α_i is a discrete variable and m_i is a hyperplane where z can be, given the discrete variable α_i and a continuous output sequence. The estimator of Theorem 4.1 is applied. Thus, $z_U(k)$ at each step k is a collection of affine linear subspaces, each given by the set of $z \in \mathbb{R}^3$ such that $M_i(k)z = (\mathbf{y}(k) - V_i(k))$, where $M_i(k) = (C(\alpha_i)', (C(f(\alpha_i))A(\alpha_i))', \ldots, (C(f^{k-1}(\alpha_i))A(f^{k-2}(\alpha_i)))')'$, and $V_i(k) = (0, C(f(\alpha_i))B(\alpha_i), \ldots C(f^{k-1}(\alpha_i))B(f^{k-2}(\alpha_i)))'$, with $\mathbf{y}(k) = (y(0), \ldots, y(k-1))'$, and α_i is such that $f^{k-1}(\alpha_i) \in [\perp, U(k)]$, for $U(k) \in \chi$ and $i \in \{1, \ldots, 5\}$. When only one α_i is left in $[\perp, U(k)]$ and the corresponding matrix $M_i(k)$ has rank equal to n, the estimator has converged. Thus, define $d_1(\perp, z_U(k)) = \sum_{i=1}^{5} \beta(M_i(k))$ where $\beta(M_i(k)) := 0$ if $f^{k-1}(\alpha_i) \notin [\perp, U(k)]$, while $\beta(M_i(k)) := (n+1) - \text{rank}(M_i(k))$, otherwise. As a consequence, when $d_1(\perp, z_U(k)) = 1$, the estimator has converged and $z(k) = M_j(k)^\dagger(\mathbf{y}(k) - V_j(k))$ for some $j \in \{1, \ldots, 5\}$, where $M_j(k)^\dagger$ is the pseudoinverse of $M_j(k)$. The value of $\beta(M_i(k))$ is the dimension of the kernel of $M_i(k)$ plus one. One is added because if $z_U(k) \ne \perp = \emptyset$ but it is equal to a singleton, the distance $d_1(\perp, z_U(k))$ cannot be zero. The continuous state estimator convergence speed depends only on the rank of $M_i(k)$ and on the discrete state estimator convergence speed. It does not depend on the specific values of B_i. Once $d_1(\perp, z_U(k)) = 1$ the state of the system is tracked (Fig. 3). In this example, the representation of the elements of $(\chi, \le)$ and of $(\mathcal{Z}_E, \le)$ involves a listing of objects: a listing of α values and a listing of linear subspaces. If $|\mathcal{U}|$ is very large (see Example 3), this choice of the partial orders renders the estimation process prohibitive.

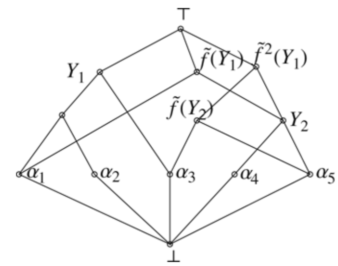

Fig. 2. Lattice $(\chi, \le)$. We have $\tilde{f}(Y_2) = \{\alpha_3, \alpha_5\}$, $\tilde{f}(Y_1) = \{\alpha_1, \alpha_4, \alpha_5\}$, and $\tilde{f}^2(Y_1) = \{\alpha_3, \alpha_4, \alpha_5\}$.

Example 2 (*Monotone System*). This example considers a system $\Sigma = \Sigma_1 \circ_c \Sigma_2$ in which Σ_1 has the same structure as in Example 1, while the continuous state update map h is defined by $z_1(k+1) = (1-\beta)z_1(k) - \beta z_2(k) + 2\beta X(\alpha(k))$ and $z_2(k+1) = (1-\lambda)z_2(k) + \lambda X(\alpha(k))$, where $\beta = 0.1$, $\lambda = 0.1$, $X(\alpha_i) := 10i$ for $i \in \{1, \ldots, 5\}$. We denote by h_2 the map that attaches to a pair (α, z_2) the value of $(1-\lambda)z_2(k) + \lambda X(\alpha(k))$. The output function g is such that $g(z, \alpha) = (g_\alpha(\alpha), g_z(\alpha, z))$, where $g_\alpha : \mathcal{U} \to \{y_1, y_2\}$ and $g_z(\alpha, z) = z_1$. The lattice $(\chi, \le)$ is shown in Fig. 2 and the system $\tilde{\Sigma}_1$ is the same as the one of Example 1. The system $\tilde{\Sigma}_2 = (\mathcal{Z}, \chi, \mathcal{Y}, \tilde{h}, \tilde{g})$ is defined as follows. We choose $\mathcal{L} = \chi \times \mathcal{Z}$, in which $\mathcal{Z} = \mathbb{R}^2$, and the ordered space $(\mathcal{Z}, \le)$ is chosen such that $(z_1^a, z_2^a) \le (z_1^b, z_2^b)$ if and only if $z_2^a \le z_2^b$. Note that this is not an ordered Banach space (condition (iii) in the definition of an ordered Banach space is not satisfied). However, this is enough for this example as we have to construct an estimator only for z_2 as z_1 is measured. Let the function $\tilde{X} : \chi \to \mathbb{R}$ be defined by $\tilde{X}(Y_1) := \max(X(\alpha_1), X(\alpha_2), X(\alpha_3)) = 30$, $\tilde{X}(Y_2) := \max(X(\alpha_3), X(\alpha_5)) = 50$, $\tilde{X}(\tilde{f}(Y_2)) = 50$, $\tilde{X}(\tilde{f}^2(Y_1)) = 50$, $\tilde{X}(\tilde{f}(Y_1)) = 50$, and $\tilde{X}(\perp) := 0$. The functions $\tilde{h}$ and $\tilde{h}_2$ are the same as h and h_2, respectively, but with $\tilde{X}$

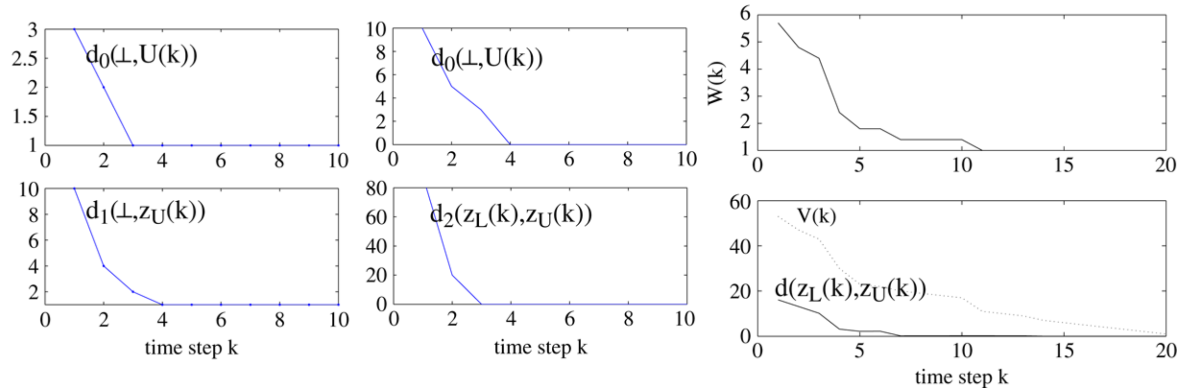

Fig. 3. Left: Estimator performance for Example 1. Middle: Estimator performance for Example 2. Right: Estimator performance for Example 3 with $N = 10$ robots per team. In the plots, $d_0(\perp, U(k)) = |[\perp, U(k)]|$.

in place of X. The function $\tilde{g}$ is defined as in Example 1, while $\tilde{g}_z = g_z$. With this choice, $\tilde{h}(z^a, w_1) \leq \tilde{h}(z^b, w_2)$ for all $(w_1, z^a) \leq (w_2, z^b)$, that is, the system is monotone. It can be shown that the system $\tilde{\Sigma} = \tilde{\Sigma}_1 \circ_c \tilde{\Sigma}_2$ is continuous state observable in two steps. Also, the observability map $M_{Y(k)}(\cdot)$ with $Y(k) = \{z_1(k), z_1(k+1)\}$ defined by $w \to \frac{1}{\beta}((1-\beta)z_1(k) - z_1(k+1) + 2\beta x_w)$ is order preserving. The estimator of Theorem 4.1 is implemented with the special structure of Eq. (9), in which z_L and z_U are the lower and upper bounds on the variable z_2 and $\tilde{h}$ is replaced by $\tilde{h}_2$. Convergence plots are shown in Fig. 3 (middle). The distance d_2 is the Euclidean distance. The representation of the elements in $\mathcal{Z}_E$ requires only n scalar numbers as $\mathcal{Z}_E = \mathcal{Z}$, and the computation of the order relation is simple. This alleviates the computation with respect to the previous example.

Example 3 (*RoboFlag Drill*). The example presented in Section 2 is revisited here. We have $\mathcal{U} = \text{perm}(N)$ and $\mathcal{Z} = \mathbb{R}^{2N}$, with output $g(z) = (z_{1,1}, \ldots, z_{N,1}) := z_1 \in \mathcal{Y} = \mathbb{R}^N$. The function $f : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$ that updates α is given by Eq. (3). Thus, $\Sigma_1 = (\mathcal{U}, \mathcal{Y}, \mathcal{U}, f, \text{id})$. The function $h : \mathcal{Z} \times \mathcal{U} \to \mathcal{Z}$ that updates the z variables is given by Eqs. (1) and (2). Thus, $\Sigma_2 = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, h, g)$. The overall system is given by $\Sigma = \Sigma_1 \circ_f \Sigma_2$. The set χ is the set of vectors in $\mathbb{N}^N$ with components less than N, and the order between any two vectors in χ is established component-wise. The extended system $\tilde{\Sigma} = \tilde{\Sigma}_1 \circ_f \tilde{\Sigma}_2$ is constructed by defining functions $\tilde{f}$ and $\tilde{h}$ as f and h, respectively, in which α is replaced by $w \in \mathbb{N}^N$. The map $\tilde{g}$ is the same as g. It can be shown that the system Σ is independently discrete state observable and that $(\tilde{\Sigma}_1, \tilde{\Sigma}, (\chi, \leq))$ is interval compatible [10]. Define the ordered space $(\mathcal{Z}, \leq)$ by choosing the positive cone K in $\mathcal{Z}$ composed by all vectors $v = (v_{1,1}, v_{1,2}, \ldots, v_{N,1}, v_{N,2})$ such that $v_{i,2} \geq 0$. The system $\tilde{\Sigma}$ is a monotone extension of Σ as the order on each $z_{i,2}$ is preserved by the dynamics in Eq. (2). The observability map defined in Section 2 is order preserving in its argument $w = (w_1, \ldots, w_N) \in \chi$ and $\tilde{\Sigma}$ is observable in two steps. The estimator in Theorem 4.1 has been implemented with the special structure of Eq. (4), in which z_L and z_U are the lower and upper bounds on z_2, respectively. The discrete state estimator is the same as the one in [10] and given in Lemma 4.1. Fig. 3 (right) illustrates the estimator performance, in which $W(k) = \sum_{i=1}^{N} |m_i(k)|$, where $|m_i(k)|$ is the cardinality of the sets $m_i(k)$ that are the sets of possible α_i for each component obtained from the sets $[L_i, U_i]$ by removing iteratively a singleton occurring at component i by all other components. When $[L(k), U(k)] \cap \text{perm}(N)$ has converged to α, we also have that $m_i(k) = \alpha_i(k)$. The distance function for $z, x \in \mathbb{R}^N$ is defined by $d(x, z) = \sum_{i=1}^{N} \text{abs}(z_i - x_i)$. The function $V(k) = \gamma(|[L(k), U(k)]|)$, defined as $V(k) := \frac{1}{2}\sum_{i=1}^{N}(x_{U_i(k)} - x_{L_i(k)})$, is always non-increasing and $d(z_L(k), z_U(k)) \leq V(k)$ for all k. Note that even if the discrete state has not converged yet, the continuous state estimation error after $k = 8$ is close to zero. This is due to the prediction-correction estimation strategy, which at each step restricts the set of all possible current continuous variable values. From Example 1 to Example 3 the computation decreases. This is due to the monotone properties of the continuous dynamics in Example 2 and in Example 3, and to the existence of a lattice $(\chi, \leq)$ with algebraic properties in Example 3. As a last remark, partial order techniques for Petri nets are another application of the state estimation theory proposed in this work (see [8] for details).

7. Conclusions

In this paper, we have proposed a methodology for the estimation of continuous and discrete variables in hybrid systems that relies on partial order structures to reduce computation. A cascade discrete-continuous state estimator has been constructed, which is the cascade interconnection of a discrete state estimator and a continuous state estimator. We have shown that the proposed techniques are general as they apply to any observable and independently discrete state observable system. The main advantage of using the partial order approach is shown when the system has some monotone properties that can be directly exploited in the estimator construction. Three examples are proposed that show the applicability of our approach and show what computational advantages can be derived in practice from its application.

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