Control of Hybrid Automata with Hidden Modes: Translation to a Perfect State Information Problem

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Abstract— In this paper, we consider the safety control problem for hybrid systems with hidden modes. In particular, we propose an approach to translate the control problem with imperfect mode information into an equivalent problem with perfect mode information. This approach is based on the notion of non-deterministic discrete information state as employed in the literature of games of imperfect information. We show that the safety control problems with imperfect information and perfect information are equivalent to each other under suitable detectability assumptions.

I. Introduction

Most of the work on safety control for hybrid systems has been focusing on the control of systems in which full state information is available [15], [10]. However, in several cases of practical interest the state is not available to the controller because of sensor or communication limitations [5], [2], [3], [17]. In this paper, we propose an approach to solve this problem in the case when the continuous state of the system is measured, while the discrete state is not measured. Furthermore, the system is subject to continuous and discrete disturbance inputs while only a continuous control input is available. This problem is found in a number of scenarios, including intent-based conflict detection and avoidance for aircrafts [14], robotic games with imperfect information [5], and semi-autonomous cooperative active safety systems to prevent vehicle collisions [16].

The safety control problem for hybrid systems with hidden modes can be viewed as a game of imperfect information between the control and the disturbance. A common approach to solving games of imperfect information is to translate the problem to an equivalent one with perfect state information [11]. In particular, [17] tackles the control of hybrid automata with hidden modes by solving an equivalent control problem with perfect state information. This new control problem was addressed by computing a capture set dependent on the mode estimate and by then designing a dynamic feedback map that maintains the flow outside this capture set. However, the conditions for the equivalence between the solved control problem and the original one with imperfect state information imposed serious restrictions on the structure of the mode estimator. In this paper, we show that this

equivalence can be proved under considerably less restrictive assumptions.

Related Work. There is a large body of literature on the safety control of hybrid automata assuming perfect state information (see, for example, [15], [10]). The solution approach for general classes of hybrid automata is based on the calculation of the backward reachable set or the uncontrollable predecessor of a bad set [8]. This set comprises all states that lead to the bad set independently of the input choice. Here, we call this set "the capture set". A feedback controller is then constructed that keeps the system state out of the capture set.

The safety control problem in the case when the set of observations is a partition of the state space was discussed by [13]. The problem was first transformed into a game of perfect information and a controller with memory was derived. The proposed algorithm can deal with a system with finite number of states. It excludes important classes of systems such as timed and hybrid automata. The safety control problem with imperfect state measurement for discrete and hybrid systems was discussed by [19]. A solution to the control problem for rectangular hybrid automata that admit a finite-state abstraction was presented. Dynamic feedback in a special class of hybrid systems with imperfect discrete state information was discussed in [2]. Dynamic control of block triangular order preserving hybrid automata under imperfect continuous state information was considered in [3] for discrete time systems and in [4] for continuous time systems.

This paper is organized as follows. Section II introduces the hybrid automaton model. In Section III, the control problem with imperfect state information is defined and an alternative problem with perfect state information is proposed. Section IV shows the equivalence between the two problems. Section V illustrates the basic concepts on an application example.

II. The System Model and Information Structure

Consider the hybrid automaton given by the tuple $H = (Q, X, U, \Delta, \Sigma, Inv, R, f)$, in which Q is a finite set of modes, X is a vector space, U is a continuous set of control inputs, Δ is a continuous set of disturbances, Σ is a finite set of disturbance events, Inv is a discrete set of silent events with $\Sigma \cap Inv = \emptyset$, $R : Q \times \Sigma \to Q$ is the discrete state update map, $f : X \times Q \times U \times \Delta \to X$ is the vector field, which is allowed to be discontinuous in the

first argument to model autonomous discrete transitions. Let $\tilde{\tau} = \bigcup_{i=0}^{N}[\tau_i, \tau'_i)]$ be a hybrid time trajectory such that $\sigma(\tau'_i) \in \Sigma$ and $\sigma(t) \in Inv$ for $t \in [\tau_i, \tau'_i)$ for all i such that $\tau_i < \tau'_i$. The ")]" parenthesis denotes that the last interval (if $N < \infty$) may be open or closed. We represent H by the following equations

$$
\begin{aligned}
q(\tau_{i+1}) &= R(q(\tau'_i), \sigma(\tau'_i)), \ \sigma(\tau'_i) \in \Sigma \qquad (1) \\
\dot{x}(t) &= f(x(t), q(t), u(t), d(t)), \ d(t) \in \Delta, \ \sigma(t) \in Inv,
\end{aligned}
$$

in which τ_i for $i \in \{0, ..., N\}$ are the times at which a discrete transition takes place and are such that $\tau_i \leq \tau'_i = \tau_{i+1}$, $q(\tau_{i+1})$ denotes the value of q after the ith transition, $q(t) := q(\sup_{\tau_i \leq t} \tau_i)$, $t \in \tilde{\tau}$ and $\sigma(t) \in Inv$, $x(0) = x_0 \in X$, $q(\tau_0) = q_0 \in Q$. We assume without loss of generality that $\tau_0 = 0$. In this model, multiple discrete transitions can occur at the same time as we can have $\tau_i = \tau_{i+1} = ... = \tau_{i+p}$ for any arbitrary $p \leq N$.

The signal $q(t)$ is a piece-wise continuous signal of time with the property that $q(t) = \lim_{\delta \to 0^+} q(t + \delta) \ \forall t \geq 0$ but $\lim_{\delta \to 0^+} q(t - \delta) \neq q(t)$ if $t = \tau_i$ for some i. Basically, at the transition time t, $q(t)$ takes the value established by the last transition occurring at time t and it maintains this value until the next transition. Since discrete transitions change only the discrete state, we have that $x(\tau_{i+1}) = x(\tau'_i)$ for all i. For convenience, we take the set Inv to be a singleton, denoted ϵ. This model is a special case of the general hybrid automaton model in standard references [12], in which there is no continuous state reset and no discrete control inputs. It may be noticed here that although the *jump* predicate is missing, the vector field is allowed to be discontinuous, which can model switches in vector field resulting from autonomous discrete transitions. An example is provided in Section V.

We denote by $\tilde{\tau}_t$ the hybrid time trajectory up to time t for $t \in \tilde{\tau}$, that is, letting $N_t := \sup\{i | \tau'_i \leq t\}$ we have $\tilde{\tau}_t = \bigcup_{i=0}^{N_t}[\tau_i, \tau'_i) \bigcup [\tau_{N_t+1}, t]$, in which $\tau_{N_t+1} = t$ if there are discrete transitions at time t. We denote by $\tilde{\sigma}_t : \tilde{\tau}_t \to \Sigma$ the discrete disturbance input signal up to time t. It forces discrete transitions at the times τ'_i and it keeps a constant value in Inv in between transitions times, that is, for $t \in [\tau_i, \tau'_i)$ for $\tau_i < \tau'_i$. We denote by $\tilde{u}_t : [0, t) \to U$ the piecewise continuous control input signal up to time t, and by $\tilde{d}_t : [0, t) \to \Delta$ the piecewise continuous disturbance input signal up to time t. Given these input signals and initial conditions $(q_0, x_0) \in Q \times X$, the discrete and continuous trajectories at any time $\tau \leq t$ are denoted by $\phi_q(\tau, q_0, \tilde{\sigma}_\tau) := q(\tau)$ and $\phi_x(\tau, (x_0, q_0), \tilde{u}_\tau, \tilde{d}_\tau, \tilde{\sigma}_\tau)$, respectively. Note that according to the definition of $q(t)$, we have that $\phi_q(0, q_0, \tilde{\sigma}_0) = q_0$ if and only if $\tau_0 < \tau_1$.

We define $\phi_x(0, (x_0, q_0), \tilde{u}_0, \tilde{d}_0, \tilde{\sigma}_0) := x_0$. The continuous trajectory $x(\tau) := \phi_x(\tau, (x_0, q_0), \tilde{u}_\tau, \tilde{d}_\tau, \tilde{\sigma}_\tau)$ satisfies $\dot{x}(\tau) = f(x(\tau), \phi_q(\tau, q_0, \tilde{\sigma}_\tau), u(\tau), d(\tau)) \ \forall \tau \leq t$. For an initial set of modes $\bar{q} \subset Q$, we denote the set of modes

reachable from $\bar{q}$ under R as $\mathcal{R}(\bar{q})$ and it is defined as

$$
\mathcal{R}(\bar{q}) := \bigcup_{q_0 \in \bar{q}} \bigcup_{t \geq 0} \bigcup_{\tilde{\sigma}_t} \phi_q(t, q_0, \tilde{\sigma}_t).
$$

Since multiple discrete transitions can occur at one time, any of these modes can be reached in no time from $\bar{q}$.

The trajectories of system (1) satisfy the following *concatenation property*. For any $t > 0$ and $t_1, t_2 > 0$ such that $t_1 + t_2 = t$, we have that $\phi_q(t, q_0, \tilde{\sigma}_t) = \phi_q(t_2, q'_0, \tilde{\sigma}'_{t_2})$ with $q'_0 = \phi_q(t_1, q_0, \tilde{\sigma}_{t_1})$, $\sigma'(\tau) = \sigma(t_1 + \tau) \ \forall \ \tau \geq 0$ and $\phi_x(t, (q_0, x_0), \tilde{u}_t, \tilde{d}_t, \tilde{\sigma}_t) = \phi_x(t_2, (q'_0, x'_0), \tilde{u}'_{t_2}, \tilde{d}'_{t_2}, \tilde{\sigma}'_{t_2})$ with $x'_0 = \phi_x(t_1, (q_0, x_0), \tilde{u}_{t_1}, \tilde{d}_{t_1}, \tilde{\sigma}_{t_1})$, $u'(\tau) = u(t_1 + \tau) \ \forall \ \tau \geq 0$, $d'(\tau) = d(t_1 + \tau) \ \forall \ \tau \geq 0$. The concatenation property implies that the value of $(q(t), x(t))$ can be uniquely determined by the values of q and x at some time $t_1 < t$ and by the values of the inputs *after* time t_1.

A. The non-deterministic discrete information state

In system (1), only x is measured while q is not. At time t, the available *information* on the system is given by the history signal $\tilde{\eta}_t := (\tilde{u}_t, \tilde{x}_t)$ with $\tilde{x}_t : [0, t] \to X$. We also denote $\eta(t) := (u(t), x(t))$. We define $\eta_0 := (\bar{q}_0, x_0)$ with $\bar{q}_0 \subset Q$ such that $q_0 \in \bar{q}_0$. This is the initial information that we have on the state of the system. On the basis of the history up to time t, we define the non-deterministic discrete information state.

Definition 1: The *non-deterministic discrete information state* at time t is the set $\bar{q}(\tilde{\eta}_t) \subset Q$ defined as

$$
\bar{q}(\tilde{\eta}_t) := \left\{
\begin{aligned}
&q \in Q \mid \exists \ q_0 \in \bar{q}_0, \ \tilde{\sigma}_t \ s.t. \ q = \phi_q(t, q_0, \tilde{\sigma}_t) \\
&\qquad\qquad\qquad\qquad \text{and } \exists \ \tilde{d}_t \ s.t. \\
&\dot{x}(\tau) = f(x(\tau), \phi_q(\tau, q_0, \tilde{\sigma}_\tau), u(\tau), d(\tau)) \\
&\qquad\qquad\qquad\qquad \text{for all } \tau < t
\end{aligned}
\right\}.
$$

Basically, $\bar{q}(\tilde{\eta}_t)$ is the set of all current modes that are compatible with the measured continuous state trajectory and with the discrete state update map R.

A consequence of this definition is that the set of all possible modes at time $t = 0$, that is, $\bar{q}(\eta_0)$, given that $\eta_0 = (\bar{q}_0, x_0)$ is given by $\bar{q}(\eta_0) = \mathcal{R}(\bar{q}_0)$ due to the possibility of multiple instantaneous transitions.

Definition 2: (*Weakly detectable modes*) We say that $q_i \in Q$ is *weakly detectable* provided for all $(u, x) \in U \times X$ there is $d \in \Delta$ such that $f(x, q_i, u, d) \neq f(x, q_j, u, \bar{d})$ for all $\bar{d} \in \Delta$ and $q_j \neq q_i$.

A mode is weakly detectable when there is a disturbance action that uniquely reveals the identity of the mode. Of course, the disturbance may choose to always play in a range so that the identity of the mode is never revealed. The property of weak detectability is useful for characterizing the possible transitions of the non-deterministic information state. In the sequel, we thus assume that all the modes in H are weakly detectable:

Assumption 1: All modes in Q are weakly detectable. If all modes are weakly detectable, there is a disturbance action at time 0^+ which leads to a measurement $\eta(0^+)$

that is compatible with only one of the modes possible at time 0. This fact and the fact that H can have multiple mode transitions at the same time leads to the following proposition.

Proposition 1: Let $\bar{q} \in 2^Q$ with $q_i \in \bar{q}$ and let $\eta_0 = (\bar{q}, x_0)$. Then, Assumption 1 implies that there is $\eta(0^+)$ such that $\bar{q}(\tilde{\eta}_{0^+}) = \mathcal{R}(q_i)$.

Because in H multiple mode transitions are possible at the same time, if $q_i \in \bar{q}(\tilde{\eta}_t)$, then all modes reachable from q_i can also be in $\bar{q}(\tilde{\eta}_t)$. Furthermore, if the measured signal $x(t)$ for all t is generated under mode q_i, then the non-deterministic discrete information state is constant for all time and equals $\mathcal{R}(q_i)$. This is formally stated by the following proposition.

Proposition 2: Let $\eta_0 = (\mathcal{R}(q_i), x_0)$ and let $\tilde{\eta}_t = (\tilde{u}_t, \tilde{x}_t)$ with $x(t) = \phi_x(t, (x_0, q_i), \tilde{u}_t, \tilde{d}_t, \epsilon)$ for all $t \geq 0$. Then, for all $\tilde{d}_t$ we have $\bar{q}(\tilde{\eta}_t) = \mathcal{R}(q_i)$ for all $t \geq 0$.

Finally, the fact that the trajectories of system H enjoy the concatenation property and the definition of $\bar{q}(\tilde{\eta}_t)$ implies also that $\bar{q}(\tilde{\eta}_t)$ enjoys the concatenation property.

The next section introduces the safety control problem for the hybrid automaton with hidden modes H.

III. The Control Problem with Imperfect Mode Information

Let $B \subset X$ be a set of continuous states to be avoided. We consider the problem of designing a dynamic feedback map that guarantees that the state never enters B for a suitable set of initial conditions. In particular, let $\pi : 2^Q \times X \to U$ and denote the closed-loop system H under such a map by H^π, whose trajectories are those of H once we set $u(t) = \pi(\bar{q}(\tilde{\eta}_t), x(t))$. We denote the x-trajectory of H^π by $\phi_x^\pi(t, (x_0, q_0), \tilde{d}_t, \tilde{\sigma}_t)$. We thus seek to determine the set of all initial conditions η_0 such that no feedback map π *with initial information* $\bar{q}(\eta_0)$ exists that can keep the trajectory $\phi_x^\pi(t, (x_0, q_0), \tilde{d}_t, \tilde{\sigma}_t)$ out of B for all time when $q_0 \in \bar{q}_0$. This set is called the *capture set* and can be written as

$$C = \bigcup_{\bar{q} \in 2^Q} \left(\bar{q}, C_{\bar{q}} \right),$$

in which $C_{\bar{q}} := \{x_0 \in X \,|\, \forall \pi, \exists q_0 \in \bar{q}, \tilde{\sigma}_t, \tilde{d}_t, t \geq 0 \text{ s.t. } \phi_x^\pi(t, (x_0, q_0), \tilde{d}_t, \tilde{\sigma}_t) \in B\}$. The set $C_{\bar{q}}$ is the set of all continuous states that enter B independently of the feedback map when the mode of the system starts in the set $\bar{q}$. This set is also referred to as *mode-dependent capture set*. Therefore, we state the problem as follows:

Problem 1: (*Control Problem with Imperfect State Information*) Determine the capture set C and the set of feedback maps π such that if $\eta(0) \notin C$, then $(\bar{q}(\eta(t)), x(t)) \notin C$ for all $t \geq 0$.

As a direct consequence of the facts that $\bar{q}(\tilde{\eta}_t)$ enjoys the concatenation property, that the open loop trajectories enjoy the concatenation property, and that π is a time-invariant map from $2^Q \times X$ to U, the trajectory of the closed loop system H^π also enjoys the concatenation property.

An immediate consequence of this fact is that if the map π keeps the trajectory of H^π outside B starting from initial information $\eta_0 = (\bar{q}_0, x_0)$, it also keeps the trajectory of H^π outside B starting from initial information $\eta_0' = (\bar{q}(\tilde{\eta}_{t_1}), x(t_1))$ for all $t_1 > 0$. This is formally stated by the following proposition.

Proposition 3: Let π be such that with $\eta_0 = (\bar{q}_0, x_0)$ we have $\phi_x^\pi(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \notin B$ for all $q_0 \in \bar{q}_0$, $\tilde{d}_t$, $\tilde{\sigma}_t$ and $t \geq 0$. Then for all $t_1 > 0$, π is such that with $\eta_0' = (\bar{q}(\tilde{\eta}_{t_1}), x(t_1))$ we have $\phi_x^\pi(t, (q_0', x(t_1)), \tilde{d}_t', \tilde{\sigma}_t') \notin B$ for all $q_0' \in \bar{q}(\tilde{\eta}_{t_1}), \tilde{d}_t', \tilde{\sigma}_t'$ and $t \geq 0$.

Note that since $\bar{q}(\eta_0) = \mathcal{R}(\bar{q}_0)$, we have that $C_{\bar{q}} = C_{\mathcal{R}(\bar{q})}$ for all modes $\bar{q} \in 2^Q$. This is formally stated in the following proposition.

Proposition 4: For all $\bar{q} \in 2^Q$, we have that $C_{\bar{q}} = C_{\mathcal{R}(\bar{q})}$.

For system H, we define the *uncontrollable predecessor operator* for a fixed mode $q_i \in Q$ and a set $S \subseteq X$ as $\text{Pre}_H(q_i, S) := \{x_0 \in X \,|\, \forall \pi, \exists \tilde{d}_t \text{ s.t. } \phi_x^\pi(t, (q_i, x_0), \tilde{d}_t, \epsilon) \in S\}$. It represents the set of all states that are taken to S independently of the feedback map whenever the mode is constant to q_i.

A. Transforming the problem to a perfect state information control problem

One of the difficulties of solving Problem 1 resides in the fact that the set $\bar{q}(\tilde{\eta}_t)$ is computed on the basis of the entire system history up to time t and keeping track of this growing history is prohibitive. We therefore translate Problem 1 to an equivalent control problem with perfect state information as performed in the theory of games with imperfect information [11].

In order to define a control problem with perfect state information, we construct a *discrete state estimate*. A discrete state estimate is a time-dependent set, denoted $\hat{q}(t) \in 2^Q$, with the properties that (i) $\bar{q}(\tilde{\eta}_t) \subseteq \hat{q}(t)$ for all $t \geq 0$; (ii) For $t_2 \geq t_1$, we have that $\hat{q}(t_2) \subseteq \mathcal{R}(\hat{q}(t_1))$. We note here that $\hat{q}(t) = Q$ for $t \geq 0$ always satisfies (i) and (ii), but in general, it is easy to construct an update law for $\hat{q}(t)$ as we show in Section V. In the case in which in (i) equality holds for all t, the estimate is said *exact*. Define the new hybrid automaton $\hat{H} = (2^Q, X, U, \Delta, Y, \hat{Inv}, \hat{R}, f)$, in which 2^Q is a new set of discrete states, Y is a set of discrete events, $\hat{Inv}$ is a set of silent events with $Y \cap \hat{Inv} = \emptyset$, $\hat{R} : 2^Q \times Y \to 2^Q$ is a discrete state transition map [1]. Let $\tilde{\tilde{\tau}} = \bigcup_{i=0}^{\hat{N}} [\hat{\tau}_i, \hat{\tau}_i']$ be a hybrid time trajectory such that $\hat{\tau}_0 = \tau_0$, $y(\hat{\tau}_i') \in Y$ and $y(t) \in \hat{Inv}$ for $t \in [\hat{\tau}_i, \hat{\tau}_i']$ for all i such that $\hat{\tau}_i < \hat{\tau}_i'$. We represent $\hat{H}$ by the following equations

$$\hat{q}(\hat{\tau}_{i+1}) = \hat{R}(\hat{q}(\tau_i'), y(\hat{\tau}_i')), \; y(\hat{\tau}_i') \in Y \qquad (2)$$
$$\dot{\hat{x}}(t) \in f(\hat{x}(t), \hat{q}(t), u(t), d(t)), \; d(t) \in \Delta, \; y(t) \in \hat{Inv}$$

[1] It is important to note here that the discrete state space of $\hat{H}$ is a subset of 2^Q. Section V provides an illustration of this.

in which we have defined $\hat{q}(t) := \hat{q}(\sup_{\hat{\tau}_i \leq t} \hat{\tau}_i) \; \forall \; t \in \tilde{\tau}$. The map $\hat{R}$ is such that $\hat{q}(t)$ is a discrete state estimate, $\hat{x}(0) = x_0$ and $\hat{q}(\hat{\tau}_0) = \bar{q}_0$. This in turn implies that (a) $\hat{R}(\hat{q}, y) \subseteq \mathcal{R}(\hat{q})$ for all $y \in Y$ and $\hat{q} \in 2^Q$ and that (b) $\hat{\tau}'_0 = \hat{\tau}_0 = 0$ and $y(\hat{\tau}'_0)$ is such that $\hat{R}(\hat{q}(\hat{\tau}'_0), y(\hat{\tau}'_0)) := \mathcal{R}(\hat{q}(\hat{\tau}'_0)) = \mathcal{R}(\bar{q}_0)$. Fix any $T > 0$, $y(t)$ derives information from $x_{\tau \in [t-T, t]}$ for $t > T$ about the values of $\dot{x}(\tau)$ for $\tau < t$ and uses this information to determine the current values of q compatible with such a derivative (see [5], [1], [6] for details).

We denote by $\tilde{\tilde{\tau}}_t$ the hybrid time trajectory of $\hat{H}$ up to time t for $t \in \tilde{\tau}$, that is, let $\hat{N}_t := \sup\{i | \hat{\tau}'_i \leq t\}$, then $\tilde{\tilde{\tau}}_t = \bigcup_{i=0}^{\hat{N}_t} [\hat{\tau}_i, \hat{\tau}'_i] \bigcup [\hat{\tau}_{\hat{N}_t+1}, t]$, in which $\tau_{\hat{N}_t+1} = t$ if there are discrete transitions at time t. We denote by $\tilde{y}_t : [0, t] \to Y$ the discrete "disturbance" input signal up to time t. It forces discrete transitions at the times $\hat{\tau}'_i$ and it keeps a constant value in $\hat{I}nv$ in between transitions times, that is, for $t \in [\hat{\tau}_i, \hat{\tau}'_i]$ for $\hat{\tau}_i < \hat{\tau}'_i$. This signal is a disturbance for system $\hat{H}$ as it is not a controlled signal but it is driven by nature's actions. Given initial conditions $(\bar{q}_0, x_0) \in 2^Q \times X$, the discrete and continuous trajectories of $\hat{H}$ at any time $\tau \leq t$ are denoted by $\phi_{\hat{q}}(\tau, \bar{q}_0, \tilde{y}_\tau) := \hat{q}(\tau)$ and $\phi_{\hat{x}}(\tau, (\bar{q}_0, x_0), \tilde{u}_\tau, \tilde{d}_\tau, \tilde{y}_\tau) := \hat{x}(\tau)$, respectively. We define $\phi_{\hat{x}}(0, (\bar{q}_0, x_0), \tilde{u}_0, \tilde{d}_0, \tilde{y}_0) := x_0$. Any continuous trajectory $\hat{x}(\tau)$ satisfies $\dot{\hat{x}}(\tau) \in f(\hat{x}(\tau), \phi_{\hat{q}}(\tau, \bar{q}_0, \tilde{y}_\tau), u(\tau), d(\tau)) \; \forall \; \tau \leq t$. We assume that $\hat{I}nv$ is a singleton and is equal to ϵ.

In system (2), the state is known as $\hat{q}(t)$ is known and $\hat{x}(t) = x(t)$ is measured. Basically, the $\hat{x}$ dynamics in (2) describes the set of dynamics of x that are compatible with the current discrete state estimate. Since $\hat{q}(t)$ is a discrete state estimate of $q(t)$, any continuous state trajectory possible in H is also possible in $\hat{H}$.

Let $\hat{\pi} : 2^Q \times X \to U$ be a feedback map. We denote the closed loop system $\hat{H}$ by $\hat{H}^{\hat{\pi}}$, the system in equations (2), in which we have set $u(t) = \hat{\pi}(\hat{q}(t), \hat{x}(t))$. The capture set for system $\hat{H}$ is given by

$$\hat{C} := \bigcup_{\bar{q} \in 2^Q} (\bar{q}, \hat{C}_{\bar{q}}),$$

in which $\hat{C}_{\bar{q}} := \{x_0 \in X | \forall \hat{\pi}, \exists \tilde{d}_t, \tilde{y}_t, t \geq 0 \; s.t. \text{ some } \phi_{\hat{x}}^{\hat{\pi}}(t, (\bar{q}, x_0), \tilde{d}_t, \tilde{y}_t) \in B\}$.

For a hybrid time trajectory such that $\hat{\tau}'_0 = \infty$, we have that $y(t) = \epsilon$ for all $t \geq 0$. We denote the corresponding continuous trajectory of $\hat{H}$ by $\hat{x}(t) = \phi_{\hat{x}}(\tau, (x_0, \bar{q}_0), \tilde{u}_\tau, \tilde{d}_\tau, \epsilon)$ and it is such that it satisfies $\dot{\hat{x}}(t) \in f(\hat{x}(t), \bar{q}_0, u(t), d(t))$ for all $t \geq 0$. We thus define for a set $S \subset X$ and $\bar{q} \in 2^Q$ the uncontrollable predecessor operator for $\hat{H}$ as $\text{Pre}(\bar{q}, S) := \{x_0 \in X | \forall \hat{\pi} \exists \tilde{d}_t, t \geq 0, \text{s.t. some } \phi_{\hat{x}}^{\hat{\pi}}(t, (x_0, \bar{q}), \tilde{d}_t, \epsilon) \in B\}$. This set represents the set of all states that are mapped to B when the mode estimate is constant to $\bar{q}$. When $\bar{q} = q_i \in q$, the Pre operator simplifies to $\text{Pre}(q_i, B) = \text{Pre}_H(q_i, B)$.

We now state the new control problem as follows.

Problem 2: (*Control Problem with Perfect State Information*) Determine the capture set $\hat{C}$ and the set of feedback maps $\hat{\pi}$ such that if $\eta(0) \notin \hat{C}$, then all $(\hat{q}(t), \hat{x}(t)) \notin \hat{C}$ for all $t \geq 0$.

This is a perfect state information problem as the hybrid state is known to the controller. This problem has been solved in [17], in which an algorithm for the computation of the mode-dependent capture sets $\hat{C}_{\bar{q}}$ was provided along with termination conditions. We recall this algorithm here for completeness. Let $2^Q = \{\hat{q}_1, ..., \hat{q}_M\}$, be the discrete state space of system $\hat{H}$ and define the tuple of sets $S_i \in 2^X$ for $i \in \{1, ..., M\}$. We define the map $G : (2^X)^M \to (2^X)^M$ as

$$G(S) := \begin{bmatrix} \text{Pre}\left(\hat{q}_1, \bigcup_{\{j | \hat{q}_j \in \hat{R}(\hat{q}_1, \mathcal{Y})\}} S_j \cup B\right) \\ \vdots \\ \text{Pre}\left(\hat{q}_M, \bigcup_{\{j | \hat{q}_j \in \hat{R}(\hat{q}_M, \mathcal{Y})\}} S_j \cup B\right) \end{bmatrix}$$

and consider the following iteration:

Algorithm 1: $S^0 := (S_1^0, S_2^0, ..., S_M^0) := (\emptyset, ..., \emptyset)$,
$S^1 = G(S^0)$
while $S^{k-1} \neq S^k$
$\qquad S^{k+1} = G(S^k)$
end.

In [17], it was shown that if Algorithm 1 terminates, the fixed point is equal to the tuple of sets $(\hat{C}_{\hat{q}_1}, ..., \hat{C}_{\hat{q}_M})$. Furthermore, [7] presented a linear complexity algorithm for computation of the Pre operator for the special case when the system dynamics are order preserving.

In this paper, we focus on determining conditions under which Problems 1 and 2 are equivalent. Specifically, we formalize the equivalence between these two problems through the following definition.

Definition 3: (*Equivalence*) We say that Problem 1 and Problem 2 are equivalent provided $C_{\bar{q}} = \hat{C}_{\bar{q}}$ for all $\bar{q} \in 2^Q$.

The next section is devoted to proving the equivalence between Problems 1 and 2.

IV. Showing the Equivalence

We show the equivalence between Problem 1 and Problem 2 by first showing that $C_{\bar{q}} \subseteq \hat{C}_{\bar{q}}$ and by then showing the reverse inclusion $C_{\bar{q}} \supseteq \hat{C}_{\bar{q}}$.

Lemma 1: For all $\bar{q} \in 2^Q$, we have that $C_{\bar{q}} \subseteq \hat{C}_{\bar{q}}$.

Proof: Proceeding by contradiction argument, assume that $x_0 \in C_{\bar{q}}$ but $x_0 \notin \hat{C}_{\bar{q}}$. Since $x_0 \in C_{\bar{q}}$, for all feedback maps π with initial non-deterministic information state $\bar{q}(\eta_0) = \mathcal{R}(\bar{q})$, there is $q_0 \in \bar{q}$, $\tilde{\sigma}_t$, $\tilde{d}_t$, and $t \geq 0$ such that $\phi_x^{\pi}(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \in B$. However, because $x_0 \notin \hat{C}_{\bar{q}}$, there is a feedback map $\hat{\pi}$ with $\hat{q}(0) = \mathcal{R}(\bar{q})$ such that for all $\tilde{d}_t$, $\tilde{y}_t$, $t \geq 0$ all flows $\phi_{\hat{x}}^{\hat{\pi}}(t, (\bar{q}, x_0), \tilde{d}_t, \tilde{y}_t) \notin B$. In particular, this is true for $\tilde{y}_t^*$ such that $y^*(t) = \epsilon$ for all $t > 0$, which implies that $\hat{q}(t) = \hat{q}(0) = \mathcal{R}(\bar{q})$ for all $t \geq 0$. Thus, there is a simple feedback map $\pi'(x) := \hat{\pi}(\hat{q}(0), x)$ such that for all $\tilde{d}_t$ and $t \geq 0$ all flows $\phi_{\hat{x}}^{\hat{\pi}'}(t, (\bar{q}, x_0), \tilde{d}_t, \tilde{y}_t^*) \notin B$ for all $t \geq 0$. A trajectory $\hat{x}(t) = \phi_{\hat{x}}^{\hat{\pi}'}(t, (\bar{q}, x_0), \tilde{d}_t, \tilde{y}_t^*)$ is in turn by definition any trajectory satisfying

$$\dot{\hat{x}}(t) \in f(\hat{x}(t), \mathcal{R}(\bar{q}), \pi'(\hat{x}(t)), d(t)). \tag{3}$$

Similarly, $x(t) = \phi_x^{\pi'}(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t)$ with $q_0 \in \bar{q}$ is any trajectory satisfying $\dot{x}(t) = f(x(t), \phi_q(t, q_0, \tilde{\sigma}_t), \pi'(x(t)), d(t))$. Since $q_0 \in \bar{q}$, it follows that any such $x(t)$ satisfies also (3) for all $\tilde{\sigma}_t$ and $\tilde{d}_t$. As a consequence, π' is such that $\phi_x^{\pi'}(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \notin B$ for all $t \geq 0$, all $\tilde{d}_t$, and all $\tilde{\sigma}_t$. This contradicts that $x_0 \in C_{\bar{q}}$. Therefore, it must be that $x_0 \in \hat{C}_{\bar{q}}$. ∎

We next focus on showing that $\hat{C}_{\bar{q}} \subseteq C_{\bar{q}}$. This is proven by first showing that $\hat{C}_{\bar{q}} \subseteq \bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B)$ and by then showing that $\bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B) \subseteq C_{\bar{q}}$. In order to show the first inclusion, we need the following structural assumption.

Assumption 2: For all $\bar{q} \in 2^Q$ we have that $\mathrm{Pre}(\bar{q}, B) = \bigcup_{q_i \in \bar{q}} \mathrm{Pre}(q_i, B)$.

This assumption is satisfied if any x that is reachable by a trajectory of system $\hat{H}$ when the mode is equal to $\hat{q} = \{q_1, ..., q_n\}$ is also reachable by a trajectory of $\hat{H}$ when the mode is equal to $\hat{q} = \{q_i\}$ for at least one $i \in \{1, ..., n\}$. This assumption can be in general checked computationally. In the special case in which the dynamics of x for $q \in \bar{q}$ and $d \in \Delta$ are order preserving, the assumption is automatically satisfied (see Proposition 5 in the Appendix). This assumption enables the proof of the following result.

Lemma 2: If Algorithm 1 terminates, under Assumptions 1 and 2 we have that $\hat{C}_{\bar{q}} \subseteq \bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B)$.

Proof: If Algorithm 1 terminates in n steps, we can write $\hat{C}_{\bar{q}} = \mathrm{Pre}\left(\bar{q}, \bigcup_{\hat{q}_{j_1} \in \hat{R}(\bar{q}, \mathcal{Y})} \mathrm{Pre}\left(\hat{q}_{j_1}, \bigcup_{\hat{q}_{j_2} \in \hat{R}(\hat{q}_{j_1}, \mathcal{Y})} \mathrm{Pre}\left(\hat{q}_{j_2}, ... \bigcup_{\hat{q}_{j_{n-1}} \in \hat{R}(\hat{q}_{j_{n-2}}, \mathcal{Y})} \mathrm{Pre}(\hat{q}_{j_{n-1}}, B)...\right)\right)\right)$. Having $\hat{q}_{j_1} \in \hat{R}(\bar{q}, \mathcal{Y})$ implies (by the definition of $\hat{R}$) that $\hat{q}_{j_1} \subseteq \mathcal{R}(\bar{q})$. Since we also have that $\hat{q}_{j_k} \subseteq \mathcal{R}(\hat{q}_{j_{k-1}})$ for $k \in \{2, ..., n-1\}$, it follows that $\hat{q}_{j_k} \subseteq \mathcal{R}(\bar{q})$. From the properties of the Pre operator (see [17]), we have that $\mathrm{Pre}(\hat{q}_{j_{n-1}}, B) \subseteq \mathrm{Pre}(\mathcal{R}(\bar{q}), B)$ and thus that $\hat{C}_{\bar{q}} \subseteq \mathrm{Pre}\left(\bar{q}, \bigcup_{\hat{q}_{j_1} \in \hat{R}(\bar{q}, \mathcal{Y})} \mathrm{Pre}\left(\hat{q}_{j_1}, \bigcup_{\hat{q}_{j_2} \in \hat{R}(\hat{q}_{j_1}, \mathcal{Y})} \mathrm{Pre}\left(\hat{q}_{j_2}, ..., \bigcup_{\hat{q}_{j_{n-2}} \in \hat{R}(\hat{q}_{j_{n-3}}, \mathcal{Y})} \mathrm{Pre}\left(\hat{q}_{j_{n-2}}(\mathrm{Pre}(\mathcal{R}(\bar{q}), B))\right)...\right)\right)\right)$. In this expression, we in turn have that $\bigcup_{\hat{q}_{j_{n-2}} \in \hat{R}(\hat{q}_{j_{n-3}}, \mathcal{Y})} \mathrm{Pre}(\hat{q}_{j_{n-2}}, (\mathrm{Pre}(\mathcal{R}(\bar{q}), B)) \subseteq \mathrm{Pre}(\mathcal{R}(\bar{q}), (\mathrm{Pre}(\mathcal{R}(\bar{q}), B)))$. By continuing substituting $\hat{q}_k$ with $\mathcal{R}(\bar{q})$, we finally obtain that $\hat{C}_{\bar{q}} \subseteq \mathrm{Pre}(\mathcal{R}(\bar{q}), \mathrm{Pre}(\mathcal{R}(\bar{q}), ..., \mathrm{Pre}(\mathcal{R}(\bar{q}), B)...))$, which by the properties of the Pre operator is equal to $\mathrm{Pre}(\mathcal{R}(\bar{q}), B)$. By Assumption 2, $\mathrm{Pre}(\mathcal{R}(\bar{q}), B) = \bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B)$, leading to $\hat{C}_{\bar{q}} \subseteq \bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B)$. ∎

The next two lemmas are intermediate steps needed to show that $\bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B) \subseteq C_{\bar{q}}$.

Lemma 3: For all $q_i \in Q$, we have that $\mathrm{Pre}(q_i, B) \subseteq C_{\mathcal{R}(q_i)}$.

Proof: Assume by contradiction argument that $x_0 \in \mathrm{Pre}(q_i, B)$ but $x_0 \notin C_{\mathcal{R}(q_i)}$. By the definition of Pre, it follows that for all maps $\pi(q_i, x)$ there is $\tilde{d}_t$ such that $\phi_x^\pi(t, (q_i, x_0), \tilde{d}_t, \epsilon) \in B$. Since the first argument of π is constant to q_i, we can define the new map $\bar{\pi}(x) := \pi(q_i, x)$. Then $x_0 \in \mathrm{Pre}(q_i, B)$ implies that for all feedback maps

$\bar{\pi}(x)$ there is $\tilde{d}_t$ such that $\phi_x^{\bar{\pi}}(t, (q_i, x_0), \tilde{d}_t, \epsilon) \in B$. If $x_0 \notin C_{\mathcal{R}(q_i)}$, there is a feedback map $\pi(\bar{q}(\tilde{\eta}_t), x)$ such that for all $q_0 \in \mathcal{R}(q_i)$, $\tilde{\sigma}_t$, $\tilde{d}_t$, it guarantees that $\phi_x^\pi(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \notin B$ for all $t \geq 0$. In particular, such a feedback map guarantees that $\phi_x^\pi(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \notin B$ for $q_0 = q_i$ and $\tilde{\sigma}_t = \epsilon$. That is, $\phi_x^\pi(t, (q_i, x_0), \tilde{d}_t, \epsilon) \notin B$ for all $t \geq 0$. By Proposition 2, we have that $\phi_x^\pi(t, (q_i, x_0), \tilde{d}_t, \epsilon)$ for all $\tilde{d}_t$ is such that $\bar{q}(\tilde{\eta}_t) = \mathcal{R}(q_i)$ for all $t \geq 0$. As a consequence, the map $\pi(\bar{q}(\eta_t), x)$ that guarantees $\phi_x^\pi(t, (q_i, x_0), \tilde{d}_t, \epsilon) \notin B$ for all $t \geq 0$ is just a simple map from x as $\bar{q}(\eta_t)$ is constant for all time. That is, we can define the new map $\bar{\pi}(x) := \pi(\mathcal{R}(q_i), x)$ so that for all $\tilde{d}_t$, it guarantees that $\phi_x^{\bar{\pi}}(t, (q_i, x_0), \tilde{d}_t, \epsilon) \notin B$ for all t. This contradicts the fact that $x_0 \in \mathrm{Pre}(q_i, B)$. Hence, if $x_0 \in \mathrm{Pre}(q_i, B)$ also $x_0 \in C_{\mathcal{R}(q_i)}$, leading to the desired result. ∎

This result is non-trivial because the feedback map involved in the definition of $\mathrm{Pre}(q_i, B)$ is a simple feedback map from x, while the one involved in the definition of $C_{\mathcal{R}(q_i)}$ has more information than only x, which derives from the current non-deterministic discrete information state.

Lemma 4: Let $q_i \in \bar{q} \in 2^Q$. Then, $C_{\mathcal{R}(q_i)} \subseteq C_{\bar{q}}$.

Proof: Let $x_0 \in C_{\mathcal{R}(q_i)}$, then for all feedback maps π with initial information $\bar{q}(\eta_0) = \mathcal{R}(q_i)$, there are $q_0 \in \mathcal{R}(q_i)$, $\tilde{\sigma}_t$, $\tilde{d}_t$, and $t \geq 0$ such that $\phi_x^\pi(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \in B$. Assume that $x_0 \notin C_{\bar{q}}$. Since $C_{\bar{q}} = C_{\mathcal{R}(\bar{q})}$ by Proposition 4, there is a feedback map π^* with initial information $\bar{q}(\eta_0) = \mathcal{R}(\bar{q})$ such that for all $q_0 \in \mathcal{R}(\bar{q})$, $\tilde{d}_t, \tilde{\sigma}_t$, $t \geq 0$ $\phi_x^{\pi^*}(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \notin B$. In particular, this must be true for all $q_0 \in \mathcal{R}(q_i) \subseteq \mathcal{R}(\bar{q})$ and for $\tilde{d}_t$ such that $d(0)$ causes $\tilde{\eta}_{0^+}$ with $\bar{q}(\tilde{\eta}_{0^+}) = \mathcal{R}(q_i)$, which exists from Proposition 1. Since $x(0^+) = x(0) = x_0$ and $\bar{q}(\tilde{\eta}_{0^+}) = \mathcal{R}(q_i)$, by Proposition 3, feedback map π^* with $\eta'_0 = (x_0, \mathcal{R}(q_i))$ is such that for all $q_0 \in \mathcal{R}(q_i)$, $\tilde{\sigma}_t$, $\tilde{d}_t$, $t \geq 0$, $\phi_x^{\pi^*}(t, (q_0, x_0), \tilde{d}_t, \tilde{\sigma}_t) \notin B$. This, however, contradicts that $x_0 \in C_{\mathcal{R}(q_i)}$. Thus if $x_0 \in C_{\mathcal{R}(q_i)}$ we must also have that $x_0 \in C_{\bar{q}}$, leading to the desired result. ∎

Note that if $\mathcal{R}(q_i) \subset \bar{q}$, this result is trivial. However, in general $\mathcal{R}(q_i) \subset \bar{q}$ is not true. Nevertheless, because of Proposition 1, we can have an instantaneous transition form $\bar{q}$ to $\mathcal{R}(q_i)$, which leads to having $C_{\mathcal{R}(q_i)} \subseteq C_{\bar{q}}$.

Lemma 5: For all $\bar{q} \in 2^Q$, we have that $\bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B) \subseteq C_{\bar{q}}$.

Proof: By Lemma 4, for all $q_i \in \bar{q}$ we have that $C_{\mathcal{R}(q_i)} \subseteq C_{\bar{q}}$. Therefore, we have that $\bigcup_{q_i \in \bar{q}} C_{\mathcal{R}(q_i)} \subseteq C_{\bar{q}}$. Employing Lemma 4 again, we obtain that for all $q'_i \in \mathcal{R}(q_i)$ we have that $C_{\mathcal{R}(q'_i)} \subseteq C_{\mathcal{R}(q_i)}$, so that $\bigcup_{q'_i \in \mathcal{R}(q_i)} C_{\mathcal{R}(q'_i)} \subseteq C_{\mathcal{R}(q_i)}$. As a consequence, we have that $\bigcup_{q_i \in \bar{q}} \bigcup_{q'_i \in \mathcal{R}(q_i)} C_{\mathcal{R}(q'_i)} \subseteq C_{\bar{q}}$, in which $\bigcup_{q_i \in \bar{q}} \bigcup_{q'_i \in \mathcal{R}(q_i)} C_{\mathcal{R}(q'_i)} = \bigcup_{q \in \mathcal{R}(\bar{q})} C_{\mathcal{R}(q)}$. Employing Lemma 3, we obtain that $\bigcup_{q \in \mathcal{R}(\bar{q})} \mathrm{Pre}(q, B) \subseteq \bigcup_{q \in \mathcal{R}(\bar{q})} C_{\mathcal{R}(q)}$, leading to the desired result. ∎

Theorem 1: If Algorithm 1 terminates, under Assumptions 1 and 2 Problem 1 and Problem 2 are equivalent.
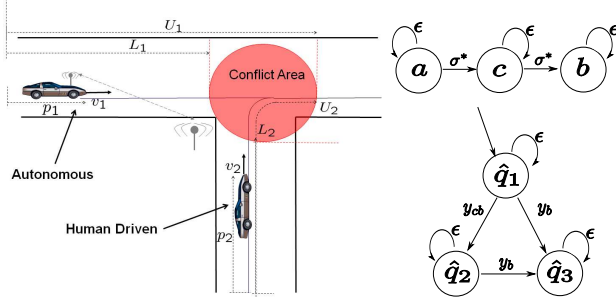
Fig. 1. **Two-vehicle Conflict Scenario**. Vehicle 1 (autonomous) is equipped with a cooperative active safety system and communicates with the infrastructure via wireless. Vehicle 2 (human-driven) is not equipped and does not communicate with the infrastructure. A collision occurs when more than one vehicle occupies the conflict area at one time. (Right) Diagram representing map R and diagram representing map $\hat{R}$.

Proof: Lemma 2 and Lemma 5 prove that for all $\bar{q} \in 2^Q$ we have that $\hat{C}_{\bar{q}} \subseteq C_{\bar{q}}$. The reverse inclusion is proven by Lemma 1. ∎

V. Application Example

As an example, consider the conflict resolution problem between two vehicles as depicted in Figure 1. The autonomous vehicle communicates with the infrastructure and has access to position and speed information about all vehicles in the intersection. The human-driven vehicle does not communicate. In order to reduce the uncertainty on its behavior, we consider a human driving model with three modes: acceleration a, coasting c, and braking b. The system can start in any of these modes and the human driven vehicle can transit from acceleration, to coasting, to braking, to model the fact that as it approaches the intersection, it may decide that it is safe to slow down [9]. This scenario can be modeled by the system $H = (Q, X, U, \Delta, \Sigma, Inv, R, f)$, in which $Q = \{a, c, b\}$, $U = [u_L, u_H]$, $\Delta = [-\bar{d}, \bar{d}]$, $\Sigma = \{\sigma^*\}$, and $R : Q \times \Sigma \to Q$ is represented in the top right diagram of Figure 1. Denoting $x = (x_1, x_2, x_3, x_4)$ with $x_1 = p_1, x_2 = v_1, x_3 = p_2, x_4 = v_2$, the unsafe set is given by $B = \{x \mid (x_1, x_3) \in [L_1, U_1] \times [L_2, U_2]\}$. The vector field f is piece-wise continuous and given by $f(x, q, u, d) = (f_1(x, u), f_2(x, q, d))$, with

$$f_1(x, u) = \begin{cases} (x_2, \alpha), & \text{if } x_2 \in (v_{min}, v_{max}) \\ (x_2, 0), & \text{if } x_2 \leq v_{min} \text{ and } \alpha < 0 \\ & \text{or } x_2 \geq v_{max} \text{ and } \alpha > 0, \end{cases} \quad (4)$$

$$f_2(x, q, d) = \begin{cases} (x_4, \beta_q + d), & \text{if } x_4 \in (v_{min}, v_{max}) \\ (x_4, 0), & \text{if } x_4 \leq v_{min} \text{ and } \beta_q + d < 0 \\ & \text{or } x_4 \geq v_{max} \text{ and } \beta_q + d > 0, \end{cases} \quad (5)$$

in which $\alpha = k_1 u - k_2 x_2^2 - k_3$ [18]. It describes the vehicles longitudinal dynamics along their paths. Here, we assume that $\beta_b < 0$, $\beta_c = 0$, and $\beta_a > 0$, with $\bar{d} < |\beta_q| < 2\bar{d}$ for $q \in \{a, b\}$. This guarantees that Assumption 1 is satisfied for x_2 and x_4 both in the open interval (v_{min}, v_{max}). This also implies that there is a confusion between modes b and c and between modes c and a. The estimator $\hat{H}$, in
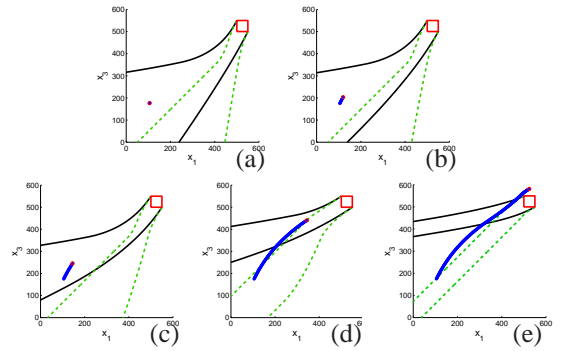

Fig. 2. In each of the plots (a)–(e), the red box represents $[L_1, U_1] \times [L_2, U_2]$. Since the sets $\text{Pre}(\hat{q}, B)_H$ and $\text{Pre}(\hat{q}, B)_L$ are four dimensional, we plot slices of these sets in the (x_1, x_3) position plane corresponding to the current speed values (x_2, x_4). The black solid lines delimit the slice of the set $\text{Pre}(\hat{q}, B)_H$ in position plane for the current speeds values (x_2, x_4). Similarly, the green dashed lines delimit the slice of the set $\text{Pre}(\hat{q}, B)_L$ in position plane for the same current speeds values (x_2, x_4). The intersection of these two slices delimits the slice of the current mode dependent capture set $\hat{C}_{\hat{q}}$ for the same current speeds values (x_2, x_4). The red circle denotes the current position x_1, x_3, while the blue trace represents the projection in the position plane of the continuous trajectory of H. Plot (a) shows the initial configuration in the position plane. Here, the current mode estimate is $\hat{q} = \{a, b, c\}$. Plot (b) shows the mode estimate switching to $\hat{q} = \{c, b\}$. Plot (c) shows the time at which the mode estimate becomes $\hat{q} = \{b\}$, so that the current mode is locked. Plot (d) shows when the continuous state hits the boundary of the current mode-dependent capture set and thus safe control is applied.

which we denote $\hat{q}_1 = \{a, b, c\}$, $\hat{q}_2 = \{c, b\}$, $\hat{q}_3 = \{b\}$, and $\hat{q}(0) = \hat{q}_1$, is uniquely defined once the set Y and map $\hat{R}$ are defined. For this sake, consider the estimate $\hat{\beta}(t) = \frac{1}{T} \int_{t-T}^{t} \dot{v}_2(\tau) d\tau$, $t \geq T$. For each possible value of $q(t)$, we compute the interval in which $\hat{\beta}(t)$ must lie. Thus, we have the following. For $q(t) = a$, we have that $|\hat{\beta}(t) - \beta_a| \leq \beta_a$; for $q(t) = c$, we have that $\hat{\beta}(t) \in [-\bar{d}, \beta_a + \bar{d}]$; and for $q(t) = b$, we have that $\hat{\beta}(t) \in [-|\beta_b| - \bar{d}, \beta_a + \bar{d}]$. Given this , we have that if $\hat{\beta}(t) \in [-|\beta_b| - \bar{d}, -\bar{d}]$ then necessarily $q(t) = b$. Similarly, if $\hat{\beta}(t) \in [-\bar{d}, 0]$ then a is not currently possible and thus we must necessarily have that $q(t) \in \{c, b\}$. As a consequence, we define $Y = \{y_{cb}, y_b, \epsilon\}$ and define for $t > T$

$$y(t) = \begin{cases} y_{cb} & \text{if } \hat{\beta}(t) \in [-\bar{d}, 0] \text{ and } \hat{q}(t^-) = \hat{q}_1 \\ y_b & \text{if } \hat{\beta}(t) \in [-|\beta_b| - \bar{d}, -\bar{d}] \text{ and } \hat{q}(t^-) \in \{\hat{q}_1, \hat{q}_2\} \\ \epsilon & \text{otherwise.} \end{cases}$$

Thus, $\hat{R}$ is represented in the bottom right diagram of Figure 1. The properties of a mode estimator are satisfied. One can easily verify that Algorithm 1 terminates and that $\hat{C}_{\hat{q}_1} = \text{Pre}(\hat{q}_1, B)$, $\hat{C}_{\hat{q}_2} = \text{Pre}(\hat{q}_2, B)$, $\hat{C}_{\hat{q}_3} = \text{Pre}(\hat{q}_3, B)$. In this case, the set of discrete modes on which $\hat{H}$ evolves is a strict subset of 2^Q given by $\hat{Q} := \{\hat{q}_1, \hat{q}_2, \hat{q}_3\}$. By virtue of Proposition 5, Assumption 2 is satisfied for all $\bar{q} \in \hat{Q}$. The sets $\text{Pre}(\hat{q}_1, B), \text{Pre}(\hat{q}_2, B)$, and $\text{Pre}(\hat{q}_3, B)$ can be easily calculated with a linear complexity discrete time

algorithm as in each mode the dynamics are given by the parallel composition of two order preserving systems and B is an interval [7]. In particular, these sets are given as $\text{Pre}(\hat{q}, B) = \text{Pre}(\hat{q}, B)_L \cap \text{Pre}(\hat{q}, B)_H$, in which $\text{Pre}(\hat{q}, B)_L = \{x \in X \mid \exists t, \tilde{d}_t \ \text{s.t. some} \ \phi_{\hat{x}}(t, (x, \hat{q}), \tilde{d}_t, u_L, \epsilon) \in B\}$ and $\text{Pre}(\hat{q}, B)_H = \{x \in X \mid \exists t, \tilde{d}_t \ \text{s.t. some} \ \phi_{\hat{x}}(t, (x, \hat{q}), \tilde{d}_t, u_H, \epsilon) \in B\}$ (see [17], [4] for more details on these computational techniques). The map $\hat{\pi}(\hat{q}, x)$ for every mode estimate $\hat{q}$ is active only when x is on the boundary of $\hat{C}_{\hat{q}}$ and in such a case it makes the continuous state slide on the boundary of $\hat{C}_{\hat{q}}$ [17], [4]. Simulation results for the closed loop system H^π are shown in Figure 2.

VI. Conclusions

In this paper, we have considered the safety control problem for hybrid systems with hidden modes. In accordance to what is performed in games of imperfect information [13], [19], [11], we translated the imperfect information control problem to a perfect information control problem. This new control problem with perfect information has been solved in our earlier work [17]. In this paper, we have focused on proving the equivalence between the two control problems under a weak detectability assumption and an assumption on the structure of the uncontrollable predecessor operator. In our future work, we will incorporate discrete control inputs and continuous state uncertainty into the model.

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Appendix

Proposition 5: Consider system H and let $X = X_1 \times X_2$ with $x_1 = (x_{1,1}, ..., x_{1,n}) \in X_1$ and $x_2 = (x_{2,1}, ..., x_{2,m}) \in X_2$. Let $\bar{q} \in 2^Q$ and assume that

(i) there are $f_1 : X_1 \times U \to X_1$, and $f_2 : X_2 \times Q \times D \to X_2$ such that $f(x, q, u, d) = (f_1(x_1, u), f_2(x_2, q, d))$ for $x_1 \in X_1$ and $x_2 \in X_2$;

(ii) there is $\bar{f}_2 : X_2 \times \mathbb{R} \to X_2$ such that we have that $\{f_2(x_2, \bar{q}, d) \mid d \in D\} = \{\bar{f}_2(x_2, \bar{d}) \mid \bar{d} \in D(\bar{q})\}$ for a suitable set $D(\bar{q}) \subset \mathbb{R}$ and the system $\dot{x}_2 = \bar{f}_2(x_2, \bar{d})$ with $\bar{d} \in D(\bar{q})$ is an order preserving system (see [4] for definition of order preserving system);

(iii) $D(\bar{q}) = \bigcup_{q \in \bar{q}} D(q)$;

(iv) $B = \{(x_1, x_2) \mid (x_{1,1}, x_{2,1}) \in [L_1, U_1] \times [L_2, U_2]\}$.

Then, $\text{Pre}(\bar{q}, B) = \bigcup_{q \in \bar{q}} \text{Pre}(q, B)$.

Proof: It is enough to show that if $(\bar{x}_1, \bar{x}_2) \in \text{Pre}(\bar{q}, B)$, then there is $q \in \bar{q}$ such that $(\bar{x}_1, \bar{x}_2) \in \text{Pre}(q, B)$. If $(\bar{x}_1, \bar{x}_2) \in \text{Pre}(\bar{q}, B)$, then for all π there is $\bar{\mathbf{d}} \in \mathcal{S}(D(\bar{q}))$ and $t \geq 0$ such that $\phi^\pi_{x_{1,1}}(t, \bar{x}_1) \in [L_1, U_1]$ and $\phi_{x_{2,1}}(t, \bar{x}_2, \bar{\mathbf{d}}) \in [L_2, U_2]$ (by (i) and (iv)). Since $D(\bar{q}) = [d_L(\bar{q}), d_H(\bar{q})]$ and the flow preserves the ordering with respect to the input by (ii), we have that $y := \phi_{x_{2,1}}(t, \bar{x}_2, \bar{\mathbf{d}}) \in [\phi_{x_{2,1}}(t, \bar{x}_2, d_L(\bar{q})), \phi_{x_{2,1}}(t, \bar{x}_2, d_H(\bar{q}))]$. Since $D(\bar{q}) = \bigcup_{q \in \bar{q}} D(q)$ by (iii) we also have that $D(\bar{q}) = \bigcup_{q \in \bar{q}} [d_L(q), d_H(q)]$. Therefore, we have that $[\phi_{x_{2,1}}(t, \bar{x}_2, d_L(\bar{q})), \phi_{x_{2,1}}(t, \bar{x}_2, d_H(\bar{q}))] = \bigcup_{q \in \bar{q}} [\phi_{x_{2,1}}(t, \bar{x}_2, d_L(q)), \phi_{x_{2,1}}(t, \bar{x}_2, d_H(q))]$. As a consequence, there is $q \in \bar{q}$ such that $y \in [\phi_{x_{2,1}}(t, \bar{x}_2, d_L(q)), \phi_{x_{2,1}}(t, \bar{x}_2, d_H(q))]$. By the continuity of the flow with respect to the input signal, we have that for all $y \in [\phi_{x_{2,1}}(t, \bar{x}_2, d_L(q)), \phi_{x_{2,1}}(t, \bar{x}_2, d_H(q))]$, there is an input signal $\bar{\mathbf{d}}' \in \mathcal{S}([d_L(q), d_H(q)])$ such that $\phi_{x_{2,1}}(t, \bar{x}_2, \bar{\mathbf{d}}') = y$. Thus, we can conclude that for all π there is $q \in \bar{q}$ and $\bar{\mathbf{d}}' \in \mathcal{S}(D(q))$ such that $\phi^\pi_{x_{1,1}}(t, \bar{x}_1) \in [L_1, U_1]$ and $\phi_{x_{2,1}}(t, \bar{x}_2, \bar{\mathbf{d}}') \in [L_2, U_2]$. This, in turn, implies that $(\bar{x}_1, \bar{x}_2) \in \text{Pre}(q, B)$. ∎