

# Safety control of piece-wise continuous order preserving systems

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**Abstract**—This paper is concerned with safety control of systems with imperfect state information and disturbance input. Specifically, we consider the class of systems whose dynamic preserve a partial ordering. We provide necessary and sufficient conditions under which a given set of initial states is steerable away from a specified bad set. Moreover, a control strategy is provided that guarantees that the bad set is avoided. Such characterization is achieved for order preserving systems while for general systems only an approximated solution is achievable. A method for implementation of the control strategy is provided and the effectiveness of the proposed method is illustrated via a numerical example and employed for obstacle avoidance of a ship.

## I. INTRODUCTION

In this paper, we consider safety control of piece-wise continuous systems with disturbance input, imperfect state information, and order preserving flow. For general hybrid systems, the safety control problem is extensively considered assuming perfect state information [1]-[6]. The control problem is addressed by computing the capture set, that is, the set of all initial states from which, with any control signal, the trajectories enter the bad set. The capture set is used to provide a static feedback law that guarantees that the bad set is avoided. The proposed methods are computationally demanding and are not guaranteed to terminate in a finite time [4]. Over-approximations of the capture set are often considered to reduce the computational burden [7], [8]. Safety control using reachability analysis is also considered [9], for nonlinear systems where the nonlinear system is approximated by a linear one and an over-approximation of reachable set is provided. Over-approximation of reachable set for linear systems is also considered via discretizing the continuous system and set-valued iterations [10]. The above approaches provide over approximation of the reachable sets.

The aforementioned works only consider systems with perfect state information where the state of the system is assumed to be available to the controller. Dynamic feedback in a special class of discrete hybrid systems with imperfect state information is considered in [12], however safety invariance is not considered. For discrete-time systems, dynamic control of block triangular order preserving hybrid automata with imperfect state information is considered in [13]. In [14], safety control results are extended to continuous time hybrid systems where the flow preserves a partial order. These results are extended to systems with input disturbances in [15]. However, the system is assumed to be in the form of the parallel composition of two decoupled systems, the

state uncertainty is assumed to be convex, and the bad set is assumed to be order preserving connected.

In this paper, we relax all these assumptions. We do not require that the system is the parallel composition of two decoupled systems. The set of initial states and bad set are just assumed to be connected. A necessary and sufficient condition for existence of a control signal that guarantees the bad set avoidance, with imperfect state information, is provided. A control strategy that guarantees the bad set avoidance is proposed. Moreover, instead of over approximation of the capture set we provide exact computation of the capture set under the assumption that the system is order-preserving. The class of order-preserving systems is widely considered from engineering and mathematical perspective. These class of systems where the flow preserves a pre-defined ordering in state space with respect to inputs and states are also called monotone systems [16]. In biology, several biological systems are shown to have the monotone property or to be composition of subsystems with monotone property [17], [18]. Transportation networks where each carrier, car or a train, moves unidirectionally according to a pre-determined path is considered as a system where a group of agents with monotone property interact [19].

In order to implement the proposed control strategy for monotone systems, an algorithm is introduced that requires forward computation of 8 trajectories over a compact interval of time corresponding to two states, two disturbance signals, and two input signals. Confining the computation over a compact interval of time guarantees termination of the algorithm in a finite time, as long as the trajectories over a finite interval are computable. Hence, the implementation algorithm is computationally as demanding as the computation of a finite-time trajectory. Therefore, the computational complexity scales linearly with the number of states. We employ the proposed control strategy in a ship steering problem where a ship must avoid an obstacle.

The paper is organized as follows. In Section II, the definitions are introduced. In Section III, the class of systems under consideration is provided and general assumptions are introduced. Problem statement is introduced in Section IV. Section V provides necessary and sufficient conditions for the set of initial states to be steerable away from the bad set. Based on the results in Section V, a control strategy is provided in Section VI that guarantees the bad set is avoided. Implementation of the proposed control strategy is elaborated in Section VII. In Section VII, numerical results are presented.

## II. DEFINITIONS

A cone  $C \subset \mathbb{R}^n$  is a set that is closed under multiplication by positive scalars and  $0 \in C$ . The sets of non-negative real numbers is denoted by  $\mathbb{R}_+$ , i.e.,  $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$ .  $\mathbb{X}$  denotes an ordered Banach space with respect to a predetermined cone.  $\|\cdot\|$  denotes the norm in Banach space  $\mathbb{X}$ . Given a Banach space  $\mathbb{X}$ ,  $x_0 \in \mathbb{X}$ , and  $\epsilon > 0$ ,  $B_\epsilon(x_0) := \{x \in \mathbb{X} \mid \|x - x_0\| < \epsilon\}$ . The set  $X^\circ \subset \mathbb{X}$  is called the set of initial states. Given a Banach space  $\mathbb{B}_u$  and  $\mathcal{U} \subset \mathbb{B}_u$ ,  $C(\mathcal{U})$  denotes the set of all piece-wise continuous functions  $R : \mathbb{R}_+ \rightarrow \mathcal{U}$ . Given a Banach space  $\mathbb{B}_u$  and  $\mathcal{U} \subset \mathbb{B}_u$ ,  $S(\mathcal{U})$  denotes the set of all measurable functions  $R : \mathbb{R}_+ \rightarrow \mathcal{U}$ . The sets  $C(\mathcal{U})$  and  $S(\mathcal{U})$  equipped with the norm  $\|f\|_\infty = \sup_{t \in \mathbb{R}_+} \|f(t)\|$  form Banach spaces denoted  $C_\infty(\mathcal{U})$  and  $S_\infty(\mathcal{U})$ , respectively. In the sequel, we drop subscript  $\infty$  to simplify notation. If  $v \in \mathbb{R}^n$ , then  $v_i$  denotes the  $i$ 'th element of the vector  $v$ . For any set  $A \subset \mathbb{R}^n$  and  $a \in \mathbb{R}$ ,  $A_{\leq(\geq)a} := \{x \in A \mid x_1 \leq (\geq) a\}$ . For  $x \in \mathbb{R}^n$  and  $A \subset \mathbb{R}^n$ ,  $d(x, A) := \inf_{y \in A} \|x - y\|$  denotes the distance of  $x$  from  $A$ . A family of non-empty compact subsets of  $\mathbb{R}^n$  is denoted by  $Com(\mathbb{R}^n)$ . For  $X \subset \mathbb{R}^n$ ,  $Cl(X)$  denotes the closure of  $X$ . Given two sets  $A, B \subset \mathbb{R}^n$ ,  $A \oplus B := \{a + b \mid a \in A \text{ and } b \in B\}$ .

A mapping  $f : \mathbb{R}^n \rightarrow Com(\mathbb{R}^n)$  is said upper-hemicontinuous at  $x_0 \in \mathbb{R}^n$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in B_\delta(x_0)$  we have that  $f(x) \subset f(x_0) \oplus B_\epsilon(0)$ . A mapping is said upper-hemicontinuous if it is upper-hemicontinuous at all points in  $\mathbb{R}^n$ . A mapping  $f : \mathbb{R}^n \rightarrow Com(\mathbb{R}^n)$  is lower-hemicontinuous at  $x_0 \in \mathbb{R}^n$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x \in B_\delta(x_0)$  we have that  $f(x_0) \subset f(x) \oplus B_\epsilon(0)$ . A mapping is said lower-hemicontinuous (upper-hemicontinuous) if it is lower-hemicontinuous (upper-hemicontinuous) at all points in  $\mathbb{R}^n$ . [11].

## III. SYSTEM CLASS

*Definition 3.1:* A system  $\Sigma$  with imperfect state information is a tuple  $\Sigma = (\mathbb{X}, \mathcal{D}, \mathcal{U}, \mathcal{M}, f, h)$ , where  $\mathbb{X} \subset \mathbb{R}^n$  is the state space,  $\mathcal{D} \subset \mathbb{R}^p$  and  $\mathcal{U} \subset \mathbb{R}^m$  are the sets of values that disturbances and control can take at each time instant, respectively, and  $\mathcal{M}$  is the set of outputs measurement,  $f : \mathbb{X} \times \mathcal{D} \times \mathcal{U} \rightarrow \mathbb{X}$  is a piecewise continuous vector field,  $h : \mathbb{X} \rightarrow 2^{\mathcal{M}}$  is the output map.

Let  $g : \mathbb{X} \rightarrow \mathcal{M}$  be a continuous output map and the mapping  $\bar{\phi} : \mathbb{R}_+ \times \mathbb{X} \times C(\mathcal{D}) \times C(\mathcal{U}) \rightarrow \mathbb{X}$  denote the flow of the system where  $C(\mathcal{U})$  is the set of control input signals and  $C(\mathcal{D})$  is the set of disturbance input signals. In addition, let  $Y := g(\bar{\phi}) : \mathbb{R}_+ \times \mathbb{X} \times C(\mathcal{D}) \times C(\mathcal{U}) \rightarrow \mathcal{M}$  denote the output of the system  $\Sigma$ .

We define the order preserving property for system  $\Sigma$  as follows

*Definition 3.2:* The system  $\Sigma$  is said to be input/output order preserving provided that

- 1) The space of disturbance signals is connected. Namely,  $C(\mathcal{D})$  is connected.
- 2) The set  $\mathcal{U}$  is partially ordered with respect to a cone  $\Delta_u \subset \mathbb{R}^m$ , i.e., for all  $u_1, u_2 \in \mathcal{U}$ ,  $u_1 \geq u_2$  if and only

if  $u_1 - u_2 \in \Delta_u$ . Moreover, there are  $u_m, u_M \in \mathcal{U}$  such that for all  $u \in \mathcal{U}$ ,  $u \geq u_m$  and  $u \leq u_M$ .

- 3) The set  $\mathcal{M}$  is partially ordered with respect to a cone  $\Delta_y \subset \mathcal{M}$ .
- 4) The flow of the system  $\Sigma$  is continuous with respect to the time. Namely, for all  $x \in \mathbb{X}$ ,  $\mathbf{u} \in C(\mathcal{U})$ , and  $\mathbf{d} \in C(\mathcal{D})$ , the mapping  $\bar{\phi}(\cdot, x, \mathbf{d}, \mathbf{u}) : \mathbb{R}_+ \rightarrow \mathbb{X}$  is continuous.
- 5) The flow of the system is continuous with respect to initial condition and disturbances, over a compact interval of time. Namely, for  $\mathbf{u} \in C(\mathcal{U})$ ,  $\epsilon > 0$ ,  $T > 0$ , there exists  $\delta > 0$  such that for all  $\mathbf{d}_0 \in C(\mathcal{D})$ , and  $x^0 \in \mathbb{X}$ , if  $\|x - x^0\| < \delta$  and  $\|\mathbf{d} - \mathbf{d}_0\| < \delta$ , then  $\sup_{t \in [0, T]} \|\bar{\phi}(t, x, \mathbf{d}, \mathbf{u}) - \bar{\phi}(t, x^0, \mathbf{d}_0, \mathbf{u})\| < \epsilon$ .
- 6) The output flow  $Y$  is an order preserving map with respect to input. Namely, for all  $\mathbf{u}_1, \mathbf{u}_2 \in C(\mathcal{U})$ , such that  $\mathbf{u}_1 \geq \mathbf{u}_2$ , we have that  $Y(t, x, \mathbf{d}, \mathbf{u}_1) \geq Y(t, x, \mathbf{d}, \mathbf{u}_2)$  for all  $x \in \mathbb{X}$ ,  $\mathbf{d} \in C(\mathcal{D})$ , and  $t \in \mathbb{R}_+$ , where the output space  $\mathcal{M}$  is partially ordered with respect to the cone  $\Delta_y$ .

Note that the order-preserving property is extensively considered and is easily verifiable as shown in [20], and reiterated in the following. The following Theorem [20].

## IV. PROBLEM STATEMENT

Under the condition that the state is not perfectly measured,  $\hat{x}(t, X^\circ, \mathbf{u}, \mathbf{z})$  denotes the set of all possible states at time  $t$  compatible with the output signal ( $\mathbf{z}$ ) measured up to time  $t$ , the control input ( $\mathbf{u}$ ) applied up to the time  $t$ , and the set of initial states, denoted by  $X^\circ$ . This set is formally defined as

$$\hat{x}(t, X^\circ, \mathbf{u}, \mathbf{z}) := \{x \in \mathbb{X} \mid \exists x_0 \in X^\circ, \mathbf{d} \in C(\mathcal{D}), \text{ s.t. } x = \bar{\phi}(t, x_0, \mathbf{d}, \mathbf{u}) \ \&\forall \tau \in [0, t], \bar{\phi}(\tau, x_0, \mathbf{d}, \mathbf{u}) \in h(\mathbf{z}(\tau))\}. \quad (1)$$

Let  $\mathbf{B} \subset \mathcal{M}$  be the bad (open) set in the output space that must be avoided. The first problem we consider is to characterize all the sets of initial sets  $X^\circ$  that are steerable away from the bad set  $\mathbf{B}$  in the output space.

Moreover, we assume the output space to be a subspace of  $\mathbb{R}^2$ , i.e.,  $\mathcal{M} \subset \mathbb{R}^2$ . The cone that orders the output space is assumed to be  $\Delta_y := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \text{ and } x_2 \leq 0\}$ .

**Problem 1:** Given system  $\Sigma$ , determine the maximal safe controlled invariant set given by

$$\mathcal{W} := \{X^\circ \subset \mathbb{X} \mid \exists \mathbf{u} \in C(\mathcal{U}), \text{ s.t. } \forall \mathbf{z} \in S(\mathcal{M}), g(\hat{x}(\mathbb{R}_+, X^\circ, \mathbf{u}, \mathbf{z})) \cap \mathbf{B} = \emptyset\}. \quad (2)$$

The second problem is to determine the control map that keeps, at each time instant, the set of all possible states away from the bad set.

**Problem 2:** Determine a control map  $K : 2^{\mathcal{M}} \rightarrow 2^{\mathcal{U}}$  such that for all output observations  $\mathbf{z} \in S(\mathcal{M})$  and  $X^\circ \in \mathcal{W}$ , we have that  $g(\hat{x}(\mathbb{R}_+, X^\circ, \mathbf{u}, \mathbf{z})) \cap \mathbf{B} = \emptyset$  if  $\mathbf{u}(t) \in K(\hat{x}(t, X^\circ, \mathbf{u}, \mathbf{z}))$ , for all  $t \in \mathbb{R}_+$ .

We solve the above two problems under the liveness assumption that is stated in the following

*Assumption 4.1:* .

- i. There exists  $\xi > 0$  such that  $f_1(x, \mathbf{d}(t), \mathbf{u}(t)) \geq \xi$  for all  $t \in \mathbb{R}_+$ ,  $\mathbf{u} \in C(\mathcal{U})$ , and  $d \in C(\mathcal{D})$ .
- ii.  $f_2(x, \mathbf{d}(t), \mathbf{u}(t)) \geq 0$  for all  $t \in \mathbb{R}_+$ ,  $\mathbf{u} \in C(\mathcal{U})$ , and  $\mathbf{d} \in C(\mathcal{D})$ .

The output of the system is the first two elements of the state  $x$ , i.e.,  $Y = g(\bar{\phi}) = (\bar{\phi}_1, \bar{\phi}_2)$ . In this paper, we have the following two more assumptions on the initial set  $X^\circ$  and the bad set  $\mathbf{B}$ .

*Assumption 4.2:* For all  $x \in X^\circ$ ,  $\mathbf{B}_{\geq x_1}$  is either empty or non-empty connected.

The above assumption guarantees that the trajectory  $Y$  intersects the bad set  $\mathbf{B}$  if and only if the output trajectory  $Y$  intersects the projected bad set  $\mathbf{B}$  in  $\mathbb{R}^2$  space.

*Assumption 4.3:* The set  $X^\circ$  is such that for all  $(b_1, b_2) \in \mathbf{B}$ ,  $\mathbb{X}_{\leq b_1}^\circ$  is either empty or non-empty connected.

The above assumption expresses that the set of all initial states that initiate a trajectory that potentially intersects the bad set is connected.

*Assumption 4.4:* .

- i. The map  $h : \mathcal{M} \rightarrow 2^{\mathbb{X}}$  is closed value, i.e., for all  $z \in \mathcal{M}$ ,  $h(z)$  is a closed set.
- ii. There exists  $z \in \mathcal{M}$ , such that  $h(z) = \mathbb{X}$ .

The above assumption expresses that there exists at least an output observation that provides no information regarding states of the system.

#### V. SOLUTION TO PROBLEM 1 : CHARACTERIZATION OF THE MAXIMAL SAFE CONTROLLED INVARIANT SET $\mathcal{W}$

In this section, we provide necessary and sufficient conditions that determine for a given set  $X^\circ$  whether there is a control signal such that for all disturbance signals, the projection of the trajectories originated from  $X^\circ$  in the output space  $\mathcal{M}$  do not enter the bad set  $\mathbf{B}$ .

Given  $\mathbf{u} \in C(\mathcal{U})$ , define the set

$$C_{\mathbf{u}} := \{x \in \mathbb{X} \mid \exists \mathbf{d} \in C(\mathcal{D}) \text{ s.t. } Y(\mathbb{R}_+, x, \mathbf{d}, \mathbf{u}) \cap \mathbf{B} \neq \emptyset\}. \quad (3)$$

The set  $C_{\mathbf{u}}$  is the set of all initial states such that there exists a disturbance signal whose corresponding trajectory intersects the bad set.

*Theorem 5.1:* Given the set of initial states  $X^\circ$ , then  $X^\circ \notin \mathcal{W}$  if and only if  $C_{\mathbf{u}_m} \cap X^\circ \neq \emptyset$  and  $C_{\mathbf{u}_M} \cap X^\circ \neq \emptyset$ .

*Proof:* Here we provide the sketch of the proof. The statement of the theorem can be rephrased as follows:  $X^\circ \notin \mathcal{W}$  if and only if  $Y(\mathbb{R}, X^\circ, C(\mathcal{D}), \mathbf{u}_m) \cap \mathbf{B} \neq \emptyset$  and  $Y(\mathbb{R}, X^\circ, C(\mathcal{D}), \mathbf{u}_M) \cap \mathbf{B} \neq \emptyset$ . The main challenge is to show the ‘‘if’’ part. Assume  $b^1, b^2 \in \mathbb{B}$ ,  $x^1, x^2 \in X^\circ$ ,  $\mathbf{d}^1, \mathbf{d}^2 \in C(\mathcal{D})$ , and  $t^1, t^2 \geq 0$  are such that  $\phi(t^1, x^1, \mathbf{d}^1, \mathbf{u}_m) = b^1$  and  $\phi(t^2, x^2, \mathbf{d}^2, \mathbf{u}_M) = b^2$ . Without loss of generality we assume  $b_1^1 \leq b_1^2$ . Consider the line  $L : x_1 = b_1^1$ . Using order preserving property, it can be shown that there exist two output trajectories initiating from the set  $X^\circ$  such that they intersect the line  $L$  at points above and below the point  $b^2$  in the plane  $\mathbb{R}^2$ . Using connectedness of  $X^\circ$  and  $C(\mathcal{D})$  continuity of the trajectories, it can be shown  $b^1 \in Y(\mathbb{R}, X^\circ, C(\mathcal{D}), \mathbf{u})$ . Hence,  $\phi(\mathbb{R}, X^\circ, C(\mathcal{D}), \mathbf{u}) \cap \mathbf{B} \neq \emptyset$ , which implies  $X^\circ \notin \mathcal{W}$ . ■

Theorem 5.1 implies that to check if the set of initial states are steerable away from the bad set, it is sufficient to only consider the behavior of the system with constant inputs  $\mathbf{u}_M$  and  $\mathbf{u}_m$ . This dramatically reduces the computational demand since it removes the need to search for all possible control inputs to determine whether a set is a member of  $\mathcal{W}$ . However, still there is a need for search over all possible disturbances. This problem will be addressed in Section VII, where the implementation problem is considered.

#### VI. SOLUTION TO PROBLEM 2: THE CONTROL STRATEGY

In this section, we introduce a control strategy that solves Problem 2. As defined in Section II, signals  $\mathbf{u}_m$  and  $\mathbf{u}_M$  adopt constant values  $u_m \in \mathcal{U}$  and  $u_M \in \mathcal{U}$ , respectively. Before introducing the control strategy, we first show some preliminary properties.

*Lemma 6.1:* Assume the set of initial states  $X^\circ$  is compact. Then, for  $\mathbf{u} \in C(\mathcal{U})$ , the set  $\bar{\phi}(t, X^\circ, C(\mathcal{D}), \mathbf{u}) := \{\bar{\phi}(t, x, \mathbf{d}, \mathbf{u}) \mid x \in X^\circ, \mathbf{d} \in C(\mathcal{D})\}$  is compact for all  $t \in \mathbb{R}$ .

*Lemma 6.2:* Given  $\mathbf{u} \in C(\mathcal{U})$ , the set valued function  $\bar{\phi}(\cdot, X^\circ, C(\mathcal{D}), \mathbf{u}) : \mathbb{R} \rightarrow \text{Com}(\mathbb{X})$  is upper-hemicontinuous.

*Lemma 6.3:* Given  $\mathbf{u} \in C(\mathcal{U})$ , the set  $C_{\mathbf{u}}$  defined in equation (3) is open in Banach space  $\mathbb{X}$ .

To characterize the control strategy, we introduce the map  $K : 2^{\mathbb{X}} \rightarrow 2^{\mathcal{U}}$  as follows:

$$K(S) = \begin{cases} u_m & \text{if } S \cap C_{\mathbf{u}_M} \neq \emptyset, S \cap C_{\mathbf{u}_m} = \emptyset \\ & \text{and } S \cap \partial C_{\mathbf{u}_m} \neq \emptyset \\ u_M & \text{if } S \cap C_{\mathbf{u}_m} \neq \emptyset, S \cap C_{\mathbf{u}_M} = \emptyset \\ & \text{and } S \cap \partial C_{\mathbf{u}_M} \neq \emptyset \\ \{u_m, u_M\} & \text{if } S \cap C_{\mathbf{u}_M} = \emptyset, S \cap C_{\mathbf{u}_m} = \emptyset, \\ & S \cap \partial C_{\mathbf{u}_m} \neq \emptyset \text{ and } S \cap \partial C_{\mathbf{u}_M} \neq \emptyset \\ U & \text{otherwise.} \end{cases} \quad (4)$$

Before introducing the control strategy, we show the compactness of  $\hat{x}$ .

*Proposition 6.1:* For all  $t \in \mathbb{R}_+$  and  $\mathbf{z} \in S(\mathcal{M})$ , the set  $\hat{x}(t, X^\circ, \mathbf{u}, \mathbf{z})$  is compact.

The following theorem proves that the control strategy provided in equation (4) prevents the flow from entering the bad set  $\mathbb{B}$ .

*Theorem 6.1:* Let the set of initial states  $X^\circ \subset \mathbb{R}^n$  be a compact set such that  $X^\circ \cap C_{\mathbf{u}_m} = \emptyset$  or  $X^\circ \cap C_{\mathbf{u}_M} = \emptyset$ . If

$$\mathbf{u}(t) \in K(\hat{x}(t, X^\circ, \mathbf{u}([0, t]), \mathbf{z})) \quad (5)$$

then  $g(\hat{x}(\mathbb{R}_+, X^\circ, \mathbf{u}, \mathbf{z})) \cap \mathbf{B} = \emptyset$ , for all  $\mathbf{z} \in C(\mathcal{M})$ .

*Proof:* Here we provide a sketch of the proof. Proceeding by contradiction, we assume there is a time  $t_1 \in \mathbb{R}_+$  such that  $\hat{x}(t_1, X^\circ, \mathbf{u}, \mathbf{z}) \cap C_{\mathbf{u}_m} \neq \emptyset$  and  $\hat{x}(t_1, X^\circ, \mathbf{u}, \mathbf{z}) \cap C_{\mathbf{u}_M} \neq \emptyset$ . Using Lemmas 6.2 and 6.3, it can be shown that there is a time  $\bar{t}$  such that  $\hat{x}(\bar{t}, X^\circ, \mathbf{u}, \mathbf{z}) \cap C_{\mathbf{u}_M} \neq \emptyset$  and  $\hat{x}(\bar{t}, X^\circ, \mathbf{u}, \mathbf{z}) \cap \partial C_{\mathbf{u}_m} \neq \emptyset$  and after the time  $\bar{t}$  the estimation set  $\hat{x}$  intersects both  $C_{\mathbf{u}_M}$  and  $C_{\mathbf{u}_m}$ . Moreover, applying control strategy (5) at the time  $\bar{t}$  prevents  $\hat{x}$  to intersect either of the two sets  $C_{\mathbf{u}_M}$  and  $C_{\mathbf{u}_m}$  for a time interval starting from  $\bar{t}$  with infinitesimally small length. This leads to a contradiction. Hence  $\hat{x}$  never intersects the two sets  $C_{\mathbf{u}_M}$  and  $C_{\mathbf{u}_m}$  simultaneously. ■

## VII. ALGORITHM IMPLEMENTATION

To implement the control strategy (5), we need to determine at each instant of time whether  $\hat{x}(t, X^o, \mathbf{u}, \mathbf{z})$  intersects the set  $C_{\mathbf{u}_m}$ , its boundary  $\partial C_{\mathbf{u}_m}$ , the set  $C_{\mathbf{u}_M}$ , and its boundary  $\partial C_{\mathbf{u}_M}$ . So far, we have considered systems that are order preserving with respect to the input as stated in Assumption 3.2-6. In this section, we consider the class of systems that not only are order preserving with respect to the input, but also are order preserving with respect to disturbance and state.

*Assumption 7.1:* For control signals  $\mathbf{u}^1$  and  $\mathbf{u}^2$ , disturbance signals  $\mathbf{d}^1$  and  $\mathbf{d}^2$  in partially ordered space  $C(\mathcal{D})$ , initial states  $x^1$  and  $x^2$  such that  $\mathbf{u}^1 \geq \mathbf{u}^2$ ,  $\mathbf{d}^1 \geq \mathbf{d}^2$ ,  $x^1 \geq x^2$ , we have that  $\bar{\phi}(t, x^1, \mathbf{d}^1, \mathbf{u}^1) \geq \bar{\phi}(t, x^2, \mathbf{d}^2, \mathbf{u}^2)$ , for all  $t \in \mathbb{R}_+$ .

In addition, we assume the cone that characterizes the ordering of the state space  $\mathbb{R}^n$  is of the following form:

$$Con = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, x_2 \leq 0, x_3 \geq (\leq) 0, \dots, x_n \geq (\leq) 0\}. \quad (6)$$

Therefore, Assumption 7.1 guarantees that the Assumption 3.2-6 is satisfied.

Hence, if  $x \geq y$  for  $x, y \in \mathbb{R}^n$ , the projection of the two vectors in  $\mathbb{R}^2$  subspace are ordered with respect to the cone defined in Section IV. The flow being order preserving with respect to the cone  $Con$  is a special case of the order-preserving property introduced in Definition 3.2. Furthermore, we assume that the closure of bad set  $\mathbf{B}$  and initial set  $X^o$  accept maximal and minimal elements with respect to the corresponding partial orderings. These assumptions are formally characterized as follows.

*Assumption 7.2:* Given the set of initial states  $X^o \subset \mathbb{R}^n$ , there are  $a_m \in X^o$  and  $a_M \in X^o$  such that for all  $a \in X^o$ , we have that  $a \geq a_m$  and  $a \leq a_M$  with partial ordering characterized by the cone  $Con$  defined in (6). Moreover, there are disturbance signals  $\mathbf{d}_m$  and  $\mathbf{d}_M$  such that for all  $\mathbf{d} \in C(\mathcal{D})$ , we have that  $\mathbf{d} \geq \mathbf{d}_m$  and  $\mathbf{d} \leq \mathbf{d}_M$ .

*Assumption 7.3:* There are  $b_m \in \partial \mathbf{B}$  and  $b_M \in \partial \mathbf{B}$  such that for all  $b \in \mathbf{B}$ , we have that  $b \geq b_m$  and  $b \leq b_M$  with partial ordering characterized by the cone  $\{(x_1, x_2) \mid x_1 \geq 0 \text{ and } x_2 \leq 0\}$ .

Note that, in the case in which the sets  $X^o$  and  $\mathbf{B}$  are rectangles, then Assumptions 7.2 and 7.3 hold. Moreover, similar to Assumption 4.3, we assume that for all  $a \in X^o$  and all  $b \in \mathbf{B}$ ,  $a_1 \leq b_1$ , i.e.,

*Assumption 7.4:*  $a_{M1} \leq b_{m1}$ .

The projection of each trajectory in  $\mathbb{R}^2$  partitions the  $\mathbb{R}^2$  space into three sets. The trajectory, the set of all points above the trajectory, and the set of all points below it. These sets are employed for the characterization of the capture set as well as for the implementation of the algorithm in the sequel.

$$\gamma^o(x, \mathbf{d}, \mathbf{u}) := \{Y(t, x, \mathbf{d}, \mathbf{u}) \mid t \in \mathbb{R}_+\} \quad (7)$$

$$\gamma^+(x, \mathbf{d}, \mathbf{u}) \quad (8)$$

$$:= \{(Y_1(t, x, \mathbf{d}, \mathbf{u}), y) \mid t \in \mathbb{R}_+ \text{ and } y > Y_2(t, x, \mathbf{d}, \mathbf{u})\}$$

$$\gamma^-(x, \mathbf{d}, \mathbf{u}) \quad (9)$$

$$:= \{(Y_1(t, x, \mathbf{d}, \mathbf{u}), y) \mid t \in \mathbb{R}_+ \text{ and } y > Y_2(t, x, \mathbf{d}, \mathbf{u})\}.$$

The following theorem proposes an implementation method to determine for a given set  $X^o$  and control signal  $\mathbf{u}$ , whether  $X^o \cap C_{\mathbf{u}} = \emptyset$ . Hence, we can determine whether  $X^o \cap C_{\mathbf{u}_m} = \emptyset$  or  $X^o \cap C_{\mathbf{u}_M} = \emptyset$ , which is essential for implementation of the control law (5).

*Theorem 7.1:* Let the set  $X^o$  be compact and  $\mathbf{u}$  be a given control signal. Then  $X^o \cap C_{\mathbf{u}} = \emptyset$  if and only if  $b_M \in Cl(\gamma^+(a_m, \mathbf{d}_m, \mathbf{u}))$  or  $b_m \in Cl(\gamma^-(a_M, \mathbf{d}_M, \mathbf{u}))$  where  $\gamma^+$ ,  $\gamma^-$  are defined in Definitions (8) and (9), respectively.

The following are the steps that should be taken to evaluate the control strategy (5).

- For a given set  $X^o$  and  $\mathbf{B}$ , identify elements  $a_m$  and  $a_M$  that satisfy Assumption 7.2 and elements  $b_m$  and  $b_M$  that satisfy Assumption 7.3.
- Take  $T \geq \frac{b_{M1}}{\xi}$ , where  $\xi$  is introduced in Assumption 4.1-i.
- For  $\mathbf{u} = \mathbf{u}_M$  and  $\mathbf{u} = \mathbf{u}_m$ , calculate  $\gamma^o(a_m, \mathbf{d}_m, \mathbf{u})$  and  $\gamma^o(a_M, \mathbf{d}_M, \mathbf{u})$  defined in (7), which determine the sets  $\gamma^+(a_m, \mathbf{d}_m, \mathbf{u})$  and  $\gamma^-(a_M, \mathbf{d}_M, \mathbf{u})$ .
- For  $\mathbf{u} = \mathbf{u}_M$  and  $\mathbf{u} = \mathbf{u}_m$ , determine if  $X^o \cap C_{\mathbf{u}} = \emptyset$  or  $X^o \cap \partial C_{\mathbf{u}} = \emptyset$  as follows
  - If  $b_M \in \gamma^+(a_m, \mathbf{d}_m, \mathbf{u})$  or  $b_m \in \gamma^-(a_M, \mathbf{d}_M, \mathbf{u})$  then  $X^o \cap C_{\mathbf{u}} = \emptyset$  and  $X^o \cap \partial C_{\mathbf{u}} = \emptyset$ .
  - If  $b_M \in \gamma^o(a_m, \mathbf{d}_m, \mathbf{u})$  or  $b_m \in \gamma^o(a_M, \mathbf{d}_M, \mathbf{u})$  then  $X^o \cap C_{\mathbf{u}} = \emptyset$  and  $X^o \cap \partial C_{\mathbf{u}} \neq \emptyset$ .
  - If  $b_M \in \gamma^-(a_m, \mathbf{d}_m, \mathbf{u})$  and  $b_m \in \gamma^+(a_M, \mathbf{d}_M, \mathbf{u})$  then  $X^o \cap C_{\mathbf{u}} \neq \emptyset$ .

## VIII. A NUMERICAL EXAMPLE

As an example to illustrate the application of the proposed algorithm, we consider the problem of steering a ship from an initial position to a desired target position, where an obstacle is avoided. The following ship model, taken from [21], is considered:

$$\begin{aligned} \dot{x}_1 &= x_5 \cos(x_3) - (r_1 x_4 + r_3 x_4^3) \sin(x_3) \\ \dot{x}_2 &= x_5 \sin(x_3) + (r_1 x_4 + r_3 x_4^3) \cos(x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -a x_4 - b x_4^3 + c u_r \\ \dot{x}_5 &= -f x_5 - W x_4^2 + u_t, \end{aligned} \quad (10)$$

where  $x_1$  and  $x_2$  are the ship position (in nautical miles (nm)) in the  $\mathbb{R}^2$  plane,  $x_3$  is the heading angle,  $x_4$  is the yaw rate, and  $x_5$  is the forward velocity. The two control inputs are: the rudder angle  $u_r$  and the propeller thrust  $u_t$ . Figure 1 represents the ship with the coordinates. The model parameters are summarized in Table I. With these parameters, the ship has a maximum speed of 0.25 nm/min = 15 knots for

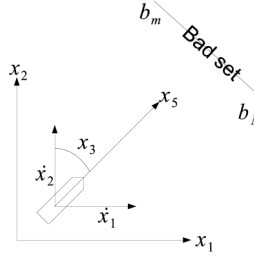


Fig. 1. Ship coordinate system. The obstacle on the path of the ship is a line segment that connects point  $b_m = (b_1, b_2) = (8, 5)$  to the point  $b_M = (b_1, b_2) = (5, 8)$ . That is, the ship is not allowed to pass over the line segment. The initial heading angle is  $x_3 = \pi/4$  and the ship initially is heading toward the target, moving toward the middle of the obstacle. The ship heading angle  $x_3$  and heading velocity  $x_4$  are measured perfectly, while the position of the ship is initially known with an uncertainty of  $\pm 0.1$  m.

a maximum thrust of  $0.215$  nm/min<sup>2</sup>. The maximal rudder angle is  $35^\circ$ , i.e.,  $|u_r| \leq u_r^m = 0.61$  rad.

Table I: Constant parameters of ship model

Parameter	value	unit
$a$	1.084	1/min
$b$	0.62	min/rad <sup>2</sup>
$c$	3.553	1/min <sup>2</sup>
$r_1$	-0.0375	nm/rad
$r_3$	0	Nm.min <sup>2</sup> /rad <sup>3</sup>
$f$	0.86	1/min
$W$	0.067	nm/rad <sup>2</sup>

With constant propeller thrust  $u_t$ , and the effect of heading velocity on the speed of the ship being negligible, the speed of the ship is assumed to be constant at  $V = 0.25$  nm/min. Therefore, for the forward velocity  $x_5$ , we have that  $x_5(t) = V$  for all  $t \geq 0$ . Moreover, according to Table I,  $r_3 = 0$ . Hence, the model is reduced to the following:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = f(x, u_r) = \begin{bmatrix} V \cos(x_3) - (r_1 x_4) \sin(x_3) \\ V \sin(x_3) + (r_1 x_4) \cos(x_3) \\ x_4 \\ -a x_4 - b x_4^3 + c u_r \end{bmatrix}. \quad (11)$$

Without loss of generality, we assume that the ship moves from the origin heading toward a target in the first orthant and therefore, the heading angle  $x_3$  is restricted to

$$\alpha \leq x_3 \leq \pi/2 - \alpha, \quad (12)$$

with  $\alpha \in [0, \pi/2]$ . This constraint is imposed to prevent the ship from getting away from the target while it avoids the obstacle. Moreover, it will be shown in the sequel that this constraint provides order preserving property once the effect of the heading velocity on position of the ship is considered as a disturbance. The obstacle on the path of the ship is a line segment that connects point  $b_m = (b_1, b_2)$  to the point  $b_M = (b_1, b_2)$ . That is, the ship is not allowed to pass over the line segment. The initial heading angle is  $x_3 = \pi/4$  and the ship initially is heading toward the

target, moving toward the middle of the obstacle. The ship heading angle  $x_3$  and heading velocity  $x_4$  are measured perfectly, while the position of the ship is initially known with an uncertainty of  $\pm 0.1$  m. The ship dynamics are characterized as  $\Sigma = (\mathbb{X}, \mathcal{D}, \mathcal{U}, \mathcal{M}, f, h)$  where  $\mathbb{X} = \mathbb{R}^4$ ,  $\mathcal{U} = \{u_r \mid |u_r| \leq u_r^m = 0.61\}$ ,  $\mathcal{D} = \emptyset$ ,  $\mathcal{M} = [-\pi, \pi] \times \mathbb{R}$ ,  $f$  is the vector-field introduced in (11), and

$$h(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = \{(x_1, x_2, x_3, x_4) \mid x_1 \in [\tilde{x}_1 - \delta x, \tilde{x}_1 + \delta x], x_2 \in [\tilde{x}_2 - \delta x, \tilde{x}_2 + \delta x], x_3 = \tilde{x}_3, \text{ and } x_4 = \tilde{x}_4\}, \quad (13)$$

where  $[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4] \in \mathcal{M}$  is the measurement. The Bad set is  $\mathbf{B} = \{b_m + (1 - \alpha)b_M \mid \alpha \in [0, 1]\}$ . The set  $\mathbf{B}$  is the line segment that connects  $b_m$  to  $b_M$ . The output map  $g$  is defined as  $Y := g(x) = (x_1, x_2)$ , where  $x$  is state and  $Y$  is the output which is the position of the ship.

We summarize the objective as follows:

- Keep the ship in the first orthant, while the constraint (12) is satisfied.
- Avoid the obstacle (the aforementioned part line), while the ship is contained in the first orthant and the constraint (12) is satisfied.

#### A. Constraining the ship in the first orthant

To constrain the ship in the first orthant, in accordance with constraint (12), we need to determine the maximal controlled invariant set,  $S_M$ , contained in the set  $\{x \in \mathbb{R}^4 \mid \alpha \leq x_3 \leq \pi/2 - \alpha\}$ . Moreover, the corresponding control law that keeps the state of the system inside  $S_M$  needs to be determined.

Considering equation (11), we have that  $\dot{x}_4 = -a x_4 - b x_4^3 + c u_r$ . Let  $x_4^m$  be such that  $-a x_4^m - b x_4^{m3} + c u_r^m = 0$ . Considering the saturation constraint  $|u_r| \leq u_r^m$ , for  $x_4 > x_4^m$  we have that

$$\dot{x}_4 = -a x_4 - b x_4^3 + c u_r < -a x_4^m - b x_4^{m3} + c u_r^m = 0.$$

Similarly, for  $x_4 < -x_4^m$ , we have that  $\dot{x}_4 > 0$ . Therefore, for all  $u_r(\cdot)$ , the set  $S_1 := \{x \mid |x_4| \leq x_4^m\}$  is an attracting invariant set and for the dynamics (11), we have  $|x_4| \leq x_4^m$ , where  $x_4^m = 0.49$  rad/sec. Hence, the maximal controlled invariant set  $S_M$  is contained in  $S_1$ . Moreover, since all states in  $S_M$  satisfy the constraint (12), the set  $S_M$  is contained in  $S := \{x \mid \alpha \leq x_3 \leq \pi/2 - \alpha, |x_4| \leq x_4^m\}$ .

The boundary of the set  $S_M$  is partially determined by the flow of the system that crosses the point  $(\alpha, 0)$  with  $u_r = u_r^m$ , called  $\Gamma_1$ , and the flow of the system that crosses the point  $(\pi/2 - \alpha, 0)$  with  $u_r = -u_r^m$ , called  $\Gamma_2$ . In the following, we introduce  $\Gamma_1$  and  $\Gamma_2$  fromally.

Let  $\bar{\phi}(t, x, u_r)$  be the flow of the system (11). Let  $x^1 = (x_3^1, -x_4^m)$  be such that  $(\alpha, 0) = (\bar{\phi}_3(\bar{t}, x^1, u_r^m), \bar{\phi}_4(\bar{t}, x^1, u_r^m))$  for some  $\bar{t} \in \mathbb{R}_+$ . Similarly, let  $x^2 = (x_3^2, x_4^m)$  be such that  $(\pi/2 - \alpha, 0) = (\bar{\phi}_3(\hat{t}, x^2, -u_r^m), \bar{\phi}_4(\hat{t}, x^2, -u_r^m))$  for some  $\hat{t} \in \mathbb{R}_+$ . By symmetry,  $\bar{t} = \hat{t}$ .

In addition, let the sets  $\Gamma_1$  and  $\Gamma_2$  be defined as follows

$$\begin{aligned} \Gamma_1 &:= (\bar{\phi}_3([0, \bar{t}], x^1, u_r^m), \bar{\phi}_4([0, \bar{t}], x^1, u_r^m)) \\ \Gamma_2 &:= (\bar{\phi}_3([0, \bar{t}], x^2, -u_r^m), \bar{\phi}_4([0, \bar{t}], x^2, -u_r^m)). \end{aligned} \quad (14)$$

Considering the above definitions, it can be shown that the control law associated with the maximal controlled invariant set  $S_M$  is given by

$$u_r(t) \in G(x_3(t), x_4(t), u(t)) = \begin{cases} u_r^m & \text{if } (x_3, x_4) \in \Gamma_1 \\ -u_r^m & \text{if } (x_3, x_4) \in \Gamma_2 \\ u(t) & \text{otherwise.} \end{cases} \quad (15)$$

### B. Avoiding the obstacle using order preserving properties of the system

In Section VIII-A, the control law (15) is introduced which guarantees the ship to stay in the first orthant and satisfy the constraint (12). System (11) under the control law (15) forms a new system with control input  $u$  as follows:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} V \cos(x_3) - (r_1 x_4) \sin(x_3) \\ V \sin(x_3) + (r_1 x_4) \cos(x_3) \\ x_4 \\ -a x_4 - b x_4^3 + c G(x_3, x_4, u) \end{bmatrix}. \quad (16)$$

According to control law (15),  $u_r(t)$  can take arbitrary values in the interval  $[-u_r^m, u_r^m]$ , as long as  $(x_3(t), x_4(t))$  is not on  $\Gamma_1 \cup \Gamma_2$ , defined in (14).

In this section, we show that system (16) is order preserving with respect to input  $u$  according to Definition 3.2-6. Consider the system (16) as the cascade of the following two systems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} V \cos(x_3) - (r_1 x_4) \sin(x_3) \\ V \sin(x_3) + (r_1 x_4) \cos(x_3) \end{bmatrix} \quad (17)$$

and

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ -a x_4 - b x_4^3 + c G(x_3, x_4, u) \end{bmatrix}, \quad (18)$$

where  $x_3$  and  $x_4$  are the outputs of system (18), inputs to system (17), and  $u$  is the input to system (18).

We will show that system (17) is order preserving according to Definition 3.2-6, when  $x_3$  is considered as a control input and  $x_4$  is considered as disturbance. Moreover, we will show that system (17) is order preserving with respect to the input according to Definition main input order-6, once  $x_4$  is considered as a disturbance in (17). Then, considering  $x_4$  as a disturbance, we conclude that system (16) is order preserving with respect to input, i.e., it satisfies Definition 3.2-6 and order-preserving with respect to fictitious disturbance  $x_4$ . Moreover, the system is order-preserving with respect to states  $x_1$  and  $x_2$ . This allows us to apply the control strategy proposed in this paper. Since  $|x_4| \leq x_4^m$ , for system (17), we consider  $x_4$  as a disturbance input  $d$  that is bounded, i.e.,  $|d| \leq x_4^m$ . System (17) then modifies to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} V \cos(x_3) - (r_1 d) \sin(x_3) \\ V \sin(x_3) + (r_1 d) \cos(x_3) \end{bmatrix} \quad (19)$$

According to [20], system (19) is order preserving with respect to state  $x_1, x_2$  and inputs  $x_3$ , and  $d$ , if all elements

of the following Jacobian matrix are non-negative

$$\begin{bmatrix} 0 & 0 & -(V \sin(x_3) + |r_1| d \cos(x_3)) & |r_1| \sin(x_3) \\ 0 & 0 & (V \cos(x_3) + |r_1| d \sin(x_3)) & -(|r_1| \cos(x_3)) \end{bmatrix}. \quad (20)$$

In system (19) and (18), we have that  $\alpha \leq x_3 \leq \pi/2 - \alpha$  and  $|d| \leq x_4^m$ . By taking  $\alpha = 4^\circ$ , we have that  $\tan(x_3) \geq \tan(\alpha) \geq \frac{|r_1| x_4^m}{V} \geq \frac{|r_1| d}{V}$ . Therefore,  $-V \sin(x_3) + |r_1| d \cos(x_3) \leq 0$ ,  $(V \cos(x_3) + |r_1| d \sin(x_3)) \geq 0$ ,  $|r_1| \sin(x_3) \geq 0$ , and  $-|r_1| \cos(x_3) \leq 0$ . Consequently, (20) holds and system (19) is order preserving with respect to state  $(x_1, x_2)$  with the cone  $\Delta$ , and inputs  $x_3$  with cone  $\{x_3 | x_3 \leq 0\}$  and input  $d$  with cone  $\{d | d \geq 0\}$ .

**Lemma 8.1:** Let  $\mathbf{u}_1$  and  $\mathbf{u}_2$  be two control signals corresponding the system (18) and  $x_3^1(\cdot) \in C(\mathbb{R})$  and  $x_3^2(\cdot) \in C(\mathbb{R})$  be the associated heading angle trajectories, respectively. If  $\mathbf{u}_1 \geq \mathbf{u}_2$  then  $x_3^1(t) \geq x_3^2(t)$  for all  $t \in \mathbb{R}_+$ . System (19) is order preserving with respect to input and state, i.e., Assumption 7.1 is satisfied, and system (18) is order preserving with respect to the input, according to Definition 3.2-6. Hence, we conclude that the cascade of systems (19) and (18), i.e., the following system

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} V \cos(x_3) - (r_1 d) \sin(x_3) \\ V \sin(x_3) + (r_1 d) \cos(x_3) \\ x_4 \\ -a x_4 - b x_4^3 + c G(x_3, x_4, u) \end{bmatrix}. \quad (21)$$

is order preserving with respect to input and disturbance. Therefore, for system (21), the output flow  $Y(t, x, u) = (\bar{\phi}_1(t, x, u), \bar{\phi}_2(t, x, u)) = (x_1(t), x_2(t))$  is order-preserving with respect to input and the cone  $\{(x_1, x_2) | x_1 \geq 0 \text{ and } x_2 \leq 0\}$ . Moreover, since  $\frac{\partial f}{\partial x_1} \equiv 0$  and  $\frac{\partial f}{\partial x_2} \equiv 0$ , system (21) is order preserving with respect to states  $x_1$  and  $x_2$ , i.e., given  $x_3^o$  and  $x_4^o$ , if  $(x_1^o, x_2^o) \geq (\bar{x}_1^o, \bar{x}_2^o)$  with respect to the cone  $\Delta$ , then  $Y(t, (x_1^o, x_2^o, x_3^o, x_4^o), u) \geq Y(t, (\bar{x}_1^o, \bar{x}_2^o, x_3^o, x_4^o), u)$  for all  $t \in \mathbb{R}_+$ .

In the ship steering problem, we have imperfect observation of the position of the ship with state estimate of the position of the ship  $\hat{x}_1$  and  $\hat{x}_2$  being  $\hat{x}_1 = [\tilde{x}_1 - 0.1, \tilde{x}_1 + 0.1]$  and  $\hat{x}_2 = [\tilde{x}_2 - 0.1, \tilde{x}_2 + 0.1]$ , while heading angle and heading velocity is perfectly measured and  $\tilde{x}_1$  and  $\tilde{x}_2$  are measured positions.

Figure 2 shows the projection of sets  $C_{\mathbf{u}_m}$  and  $C_{\mathbf{u}_M}$  onto the  $\mathbb{R}^2$  space  $(x_1, x_2)$  for  $x_4 = 0$  and  $x_3 = \pi/4$ . The box represents the position estimate of the ship. The ship keeps constant heading angle  $x_3 = \pi/4$  heading the middle of the obstacle towards the target which is located beyond obstacle. According to control strategy (4) for a singleton state set, the ship applies  $\mathbf{u}(t) = \mathbf{u}_M(t) = -u_r^m$ , once at the discrete-time instant  $nT$   $x(nT) \notin C_{\mathbf{u}_m} \cup C_{\mathbf{u}_M}$  and the predicted state  $x((n+1)T) \in C_{\mathbf{u}_M} \cap C_{\mathbf{u}_m}$  with  $u_M = -u_r^m = -0.61$  and  $u_m = u_r^m = 0.61$ . Detecting if the state belongs to the sets  $C_{\mathbf{u}_m}$  or  $C_{\mathbf{u}_M}$  is performed by forward propagation of the trajectories for  $u(t) = \pm u_r^m = \pm 0.61$  and  $d = \pm x_4^m = \pm 0.49$  and construction of  $\gamma^+$  and  $\gamma^-$  introduced in (8) and (9), respectively. As shown in Figure 2 (a),  $\hat{x}(t) \cap C_{\mathbf{u}_m} \neq \emptyset$ ,  $\hat{x}(t) \cap \partial C_{\mathbf{u}_M} \neq \emptyset$ , and  $\hat{x}(t) \cap C_{\mathbf{u}_M} = \emptyset$ . Therefore, according to control law (5),

the control  $\mathbf{u}(t) = \mathbf{u}_M(t)$  is employed. Figure 2 (b) shows that as the ship moves the set of state estimate is tangent to the capture set  $C_{\mathbf{u}_M}$ . At each sample time instant  $t_n$ , the state estimate  $\hat{x}$ , is constructed by intersection of the evolution of the previous state estimate set  $\hat{x}(t_{n-1})$ , i.e.,  $L_n := Y(t_n - t_{n-1}, \hat{x}(t_{n-1}), \mathbf{u}, \mathbf{z})$ , and  $h(\mathbf{z}(t_n))$ . At each time instant  $t_n$ ,  $\mathbf{z}(t_n)$  is uniformly randomly chosen in the vicinity of  $L_n$  such that  $h(\mathbf{z}(t_n)) \cap L_n \neq \emptyset$ .  $\hat{x}(t_{n-1})$  is a rectangle that is characterized at each time instant by its center, width and length.

The sets  $C_{\mathbf{u}_M}$  and  $C_{\mathbf{u}_m}$ , are calculated according to maximum and minimum values of the fictitious disturbance  $d = x_4$ , i.e.,  $d = 0.49 \text{ rad/sec}$  or  $d = -0.49 \text{ rad/sec}$  in the modified dynamics (21). Hence, the actual dynamic may not act according to the extreme cases. Therefore, in Figure 2 (c), we observe that the estimate set is close to  $\partial C_{\mathbf{u}_M}$  but does not intersect it. Figure 2 (d), shows the time when the ship passes the obstacle.

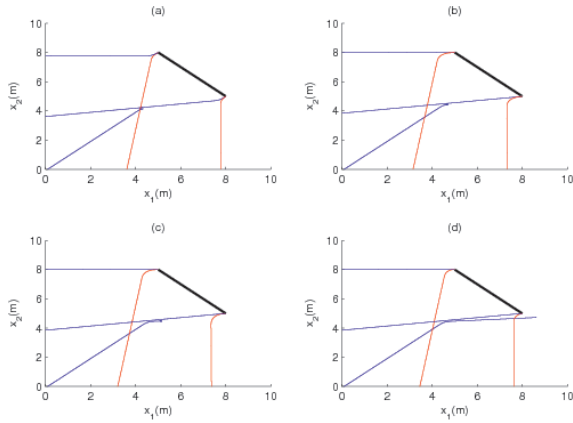


Fig. 2. Ship trajectory, slices of sets  $C_{\mathbf{u}_M}$  and  $C_{\mathbf{u}_m}$  corresponding different heading angles and yaw velocities, and obstacle (the black part line). The bad set  $\mathbf{B}$  is the black part line. Four different time instants are depicted: (a) shows the time instant where  $\hat{x}(t) \cap C_{\mathbf{u}_m} \neq \emptyset$ ,  $\hat{x}(t) \cap \partial C_{\mathbf{u}_M} \neq \emptyset$ , and  $\hat{x}(t) \cap C_{\mathbf{u}_M} = \emptyset$  and according to control law (5),  $\mathbf{u}_M$  is applied. (b) and (c)  $\hat{x}(t)$  is close to the set  $C_{\mathbf{u}_M}$  but  $\hat{x}(t) \cap \partial C_{\mathbf{u}_M} = \emptyset$ ,  $\hat{x}(t) \cap C_{\mathbf{u}_M} = \emptyset$  and  $\hat{x}(t) \cap C_{\mathbf{u}_m} \neq \emptyset$ . (d) shows when the ship has passed the obstacle.

## IX. CONCLUSION

This paper considers the safety control of systems with imperfect state information, with disturbance input, and with order preserving flow. Under certain assumptions, the set of all initial state uncertainties that are steerable away from a bad set is fully characterized. Assuming that the set of initial state uncertainty complies with the aforementioned assumptions, a control strategy is provided that guarantees that no possible trajectory intersects the bad set in the presence of disturbances and imperfect information. Consequently, a method for implementation of the control strategy is provided and the effectiveness of the proposed method is illustrated via a numerical example. In this example, the method is employed for obstacle avoidance of a ship in which imperfect information about the position of the ship is available.

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